Measuring with unscaled pots - algorithm versus chance

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Motto: *I cook every chance in my pot. And only when it hath been quite cooked do I welcome it as my food.*
Friedrich Nietzsche

Abstract. The central focus of this paper is on the following problem:

Consider three unscaled pots, with volumes a, b and $c > max\{a, b\}$ liters, where $a, b \in \mathbb{N}^*$. Initially the third pot is filled with water and the other ones are empty. Characterize all quantities that can be measured using these pots.

In the first part of the paper we solve this problem by using the motion of a billiard ball on a special parallelogram shaped table. In the second part we generalize the initial problem for $n+1$ pots ($n \in \mathbb{N}, n \geq 2$) and we give an algorithmic solution to this problem. This solution is also based on the properties of the orbit of a billiard ball. In the last part we present our observations and conclusions based on a problem solving activity related to this problem.

The initial problem for 3 pots is mentioned in [2] (The three jug problem on page 89), but the solution is not detailed and the general case (with several pots) is not mentioned. The visualization we use is a key element in developing the proof of our results, so the proof can be viewed as a good example of visual thinking used in arithmetic (see [3], [1]).

Keywords: linear diophantine equation, visualization, billiard

Mathematical Subject Classification: 97F60, 97E50, 97B20

Introduction

The following problem was solved by Siméon Denis Poisson using graphs in the 18^{th} century ([7]):

A man has 12 pt[3] wine and he wants to give to a neighbor 6 pints but he has only a 5 pt and an 8 pt empty pot. How can he measure 6 pt to the 8 pt pot?

Poisson's idea was to represent the possible states of the pots as vertices of a graph while every possible filling corresponds to an oriented edge in this graph. In this way starting from the initial state $(12, 0, 0)$ one can obtain all possible states as follows. For the first filling we have two possibilities, so we obtain two possible states: $(4, 8, 0)$ and $(7, 0, 5)$. From these states we can obtain the states $(0, 8, 4), (4, 3, 5), (0, 7, 5), (7, 5, 0)$ and so on.

In figure 1 we illustrated a few vertices and edges of this graph (at each level we put the states that were not already in the graph and for the simplicity we omitted the backward edges between different branches).

However this representation also leads to an algorithmic solution (the generation of this graph level by level) we need a different approach in order to solve some general problems:

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[3]1 pt (pint) is equivalent to 568.26125 ml

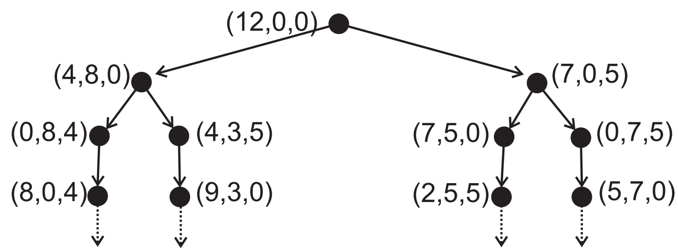


Figure 1: Poisson's representation

a) Consider three unscaled pots, with volumes a, b and $c > max\{a, b\}$ liters, where $a, b \in \mathbb{N}^*$. Initially the third pot is filled with water and the other ones are empty. Characterize all the quantities which can be measured with these pots.

b) Consider $n+1$ unscaled pots, with volumes $a_1, a_2, \ldots, a_n$ and a_{n+1} liters, where $a_i \in \mathbb{N}^*, 1 \leq i \leq n+1$ and $a_{n+1} \geq \max\limits_{1 \leq i \leq n} a_i$. Initially the largest pot is filled with water and the other ones are empty. Characterize all the quantities which can be measured with these pots.

Regarding a) in [2] the author states that "Clearly, such a problem (with $c = a + b$) can be solved whenever the integers a and b are coprime", but there is no proof of this assertion. Hence our first aim is to give a detailed answer to a) and then to extend our argument to the general case. After we clarify the mathematical background we present a problem solving activity which was designed in order to investigate the solving mechanisms/algorithms our students use in handling such problems. More precisely we point out that most of our students use a "trial-error" type random algorithm (they are simply filling randomly chosen pots and they only care about avoiding previous states). Moreover we designed also a computer simulation which solves the problem by the same random algorithm (in each step it randomly chooses two pots such that by filling from the first to the second none of the previous states appears) and we observed that in all cases the solution can be obtained in this way. This fact implies that similar problems do not really measure the combinative skills of our students but their persistence, patience and vigilance.

A model, an algorithmic approach and some further mathematical background

Consider an $a \times b$ parallelogram in the lattice generated by a parallelogram with sides of length 1 and having an angle of 60°. This is our billiard table and we shall study the motion of a billiard ball which starts from the point $O(0,0)$ and moves along the edge OA (where $A(a,0)$).

As described in [2] the motion of the billiard ball on this special table gives a possible filling sequence with the pots a, b, c. For a better understanding label the diagonals as in figure 2 and to each lattice point P assign the coordinates of the lattice point and the number of the diagonal passing through P. The assigned numbers correspond to the quantity of water in the pots. The starting point corresponds to the state $(0,0,c)$, the point

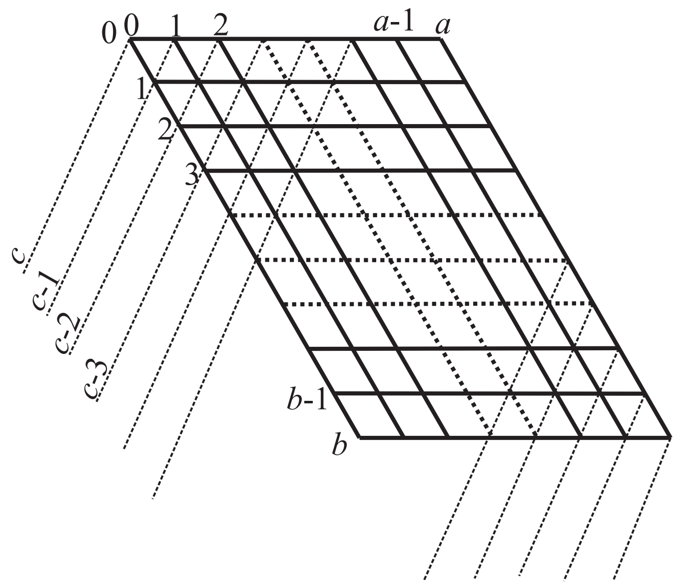


Figure 2: The billiard table

A to the state $(a, 0, c - a)$ and so on. Due to the construction of the table the ball moves along the grid lines and the diagonals and each collision point on the boundary corresponds to an achievable state of the three pots. For a better understanding we considered $a = 4$, $b = 7$ and $c = 11$ and we described the orbit of the billiard ball on figure 3. In this case in every pot it can be measured every non negative integer quantity that does not exceed the maximum capacity of the pot. Geometrically this fact means that the orbit passes through every lattice point on the boundary. In the next section we prove the following

Theorem 1. *If $c = a + b$ and $d = gcd(a, b)$ the orbit of the billiard ball (on the corresponding table) passes through a lattice point (x, y) on the boundary if and only if $d|x$ and $d|y$ ($gcd(a, b)$ denotes the greatest common divisor of a and b)*

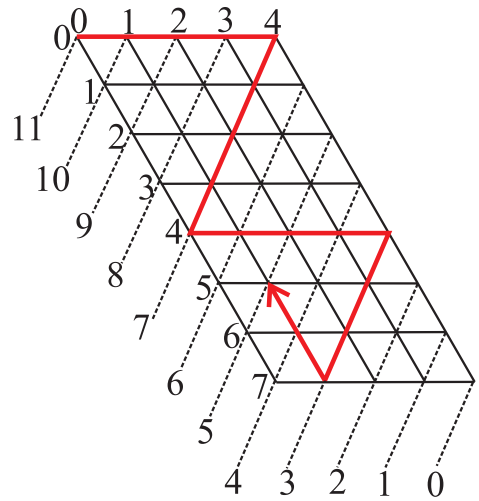
Remark 1. *If $d = 1$, the orbit passes through all the lattice points on the boundary.*

Remark 2. *If $d = gcd(a, b)$, every quantity which can be measured (without throwing water away) must be divisible by d, hence the above theorem gives an answer to problem a).*

From an algorithmic point of view either we use the billiard table to generate the sequence of states or we can formulate the following very simple strategy:

- if possible fill from a to b;
- if b is full, fill from b to c;
- if none of the previous steps is possible then fill from c to a;

This algorithm generates the same sequence of states as the motion of the ball.



(0,0,11)	(2,0,9)
(4,0,7)	(0,2,9)
(0,4,7)	(4,2,5)
(4,4,3)	(0,6,5)
(1,7,3)	(4,6,1)
(1,0,10)	(3,7,1)
(0,1,10)	(3,0,8)
(4,1,6)	(0,3,8)
(0,5,6)	(4,3,4)
(4,5,2)	(0,7,4)
(2,7,2)	(4,7,0)

Figure 3: The orbit of a ball and the states of the pots

Remark 3. *Suppose $a < b$ and $d = gcd(a, b)$. If we can measure d liters in the second pot (with b liters) by filling from the first pot (with a liters) to second one x times and by emptying the second pot y times, than $ax - by = d$, so our filling algorithm gives an algorithmic solution of the linear diophantine equation $ax - by = d$. Unfortunately the converse is not obvious. If we have the solutions of the equation $ax - by = 1$, we still need an algorithm to obtain the desired quantities in our pots. Hence the measuring problem is not equivalent with the diophantine equation.*

If we have more pots the problem seems to be more complicated, but in fact we can use the same visualization because in a filling step only 2 pots are involved, so if $(x_1, x_2, \ldots, \ldots x_n, x_{n+1})$ describes the state of pots before a filling operation and $(x'_1, x'_2, \ldots, x'_n, x'_{n+1})$ after this operation, then exactly 2 components are changed. This implies that even if we use a multidimensional visual representation (an n dimensional parallelogram), the transformations will be represented on some 2 dimensional faces, so we can also operate these transformations in the plane. If $a_1, a_2, \ldots, a_n, a_{n+1}$ represents the volumes of the pots and $d_j = gcd(a_1, a_2, \ldots, a_j)$, $j \geq 2$ then we have

$$d_3 = gcd(a_1, a_2, a_3) = gcd(gcd(a_1, a_2), a_3) = gcd(d_2, a_3)$$
$$d_4 = gcd(a_1, a_2, a_3, a_4) = gcd(gcd(a_1, a_2, a_3), a_4) = gcd(d_3, a_4)$$

and generally

$$d_{j+1} = gcd(d_j, a_{j+1}), \quad j \geq 2.$$

Consider the parallelograms with side lengths (a_1, a_2), (a_2, a_3), $(a_3, a_4), \ldots, (a_{n-1}, a_n)$ and (a_n, a_1) all having an angle of $60°$ as shown in figure 4. We consider the motion of a billiard ball on the first table with side lengths a_1 and a_2 and we mark each collision point on the common side of the first two tables. From each such point we consider the motion of a billiard ball on the second table and we mark each collision point on the common side of the second and third tables and so on. For $1 \leq j \leq n-1$ denote by S_j the common side of the j^{th} and $(j+1)^{th}$ tables. The length of S_j is a_{j+1}. We are interested in characterizing all marked points on the segments $S_1, S_2, S_3, \ldots, S_{n-1}$. In order to obtain this characterization we rephrase our Theorem 1 as follows:

Theorem 2. *If d' is a divisor of b and we consider all the orbits (of a billiard ball on the table with side lengths a and b) starting from the points $(0, kd')$, where $k \in \mathbb{N}$ and $kd' \leq b$, then these orbits will contain the lattice point (x, y) from the boundary if and only if $d|x$ and $d|y$ where $d = (d', a)$.*

This theorem guaranties that on every segment S_j we mark exactly the points whose coordinates are multiples of d_{j+1}, hence on the segment S_{n-1} (with length a_n) we mark all points whose coordinates are multiples of $d = gcd(a_1, a_2, \ldots, a_n)$. Due to the symmetry this can be extended to all segments, which means that in each pot we can measure a quantity x if and only if $d|x$ and x does not exceed the capacity of the pot.

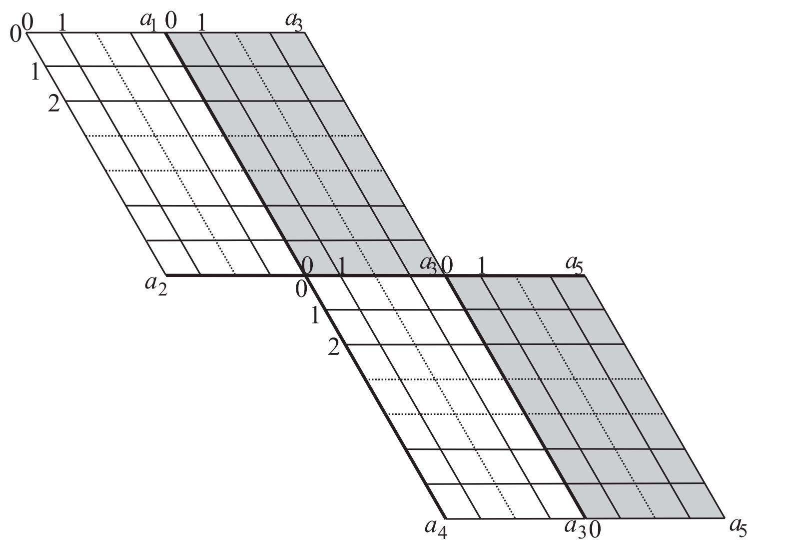


Figure 4: The unfolded faces

Due to the previous argumentation we have the following theorems:

Theorem 3. *Consider three unscaled pots, with volumes a, b and $c > max\{a, b\}$ liters, where $a, b \in \mathbb{N}^*$. Initially the third pot is filled with water and the other ones are empty.*

- *If $c = a + b$ and $(a, b) = d$, then in the pot with volume a we can measure $0, 1 \cdot d, 2 \cdot d, \ldots, a - d, a$ liters, in the pot with volume b we can measure $0, 1 \cdot d, 2 \cdot d, \ldots, b - d, b$ liters and in the pot with volume c we can measure $0, 1 \cdot d, 2 \cdot d, \ldots, c - d, c$ liters.*

- *If $c > a + b$ and $(a, b) = d$, then in the pot with volume a we can measure $0, 1 \cdot d, 2 \cdot d, \ldots, a - d, a$ liters, in the pot with volume b we can measure $0, 1 \cdot d, 2 \cdot d, \ldots, b - d, b$ liters and in the pot with volume c we can measure $c - a - b, c - a - b + 1 \cdot d, c - a - b + 2 \cdot d, \ldots, c - d, c$ liters.*

Theorem 4. *Consider $n + 1$ unscaled pots with volumes $a_1, a_2, \ldots, a_n$ and a_{n+1}, where $a_1, a_2, \ldots, a_n \in \mathbb{N}^*$ and denote by d the greatest common divisor of $a_1, a_2, \ldots, a_n$. Initially* *the last pot is filled with water. If* $a_{n+1} \geq \sum_{j=1}^{n} a_j$*, then for each* $j \in \{1, 2, \dots n\}$ *in the pot with volume* a_j *we can measure* $0, 1 \cdot d, 2 \cdot d, \dots, a_j - d, a_j$ *liters and in the pot with volume* a_{n+1} *we can measure* $c, c+d, c+2d, \dots, a_{n+1} - d, a_{n+1}$ *liters, where* $c = a_{n+1} - \sum_{j=1}^{n} a_j$.

Remark 4. *We have created a Matlab program which illustrates the motion of the billiard ball and the corresponding states of the pots. This can be viewed at*

http://www.math.ubbcluj.ro/~andrasz/filling/animation

Proofs

In this section we prove the asserted theorems using the billiard ball's motion and some basic number theoretic properties.

Proof of theorem 1. The key observation in our proof is a relation between the coordinates of the successive upper and the lower impact points. If we have an impact point on the upper boundary segment with coordinates $(a-x, 0)$ and the next impact point on the lower boundary segment has coordinates $(a-y, b)$, then y is the remainder obtained dividing $x+b$ with a (see figure 5). Due to this observation the coordinates of the impact points on the lower boundary segment are the remainders obtained dividing $b, 2b, 3b, \dots, (a_1 - 1)b, a_1 b$ with a, where $a = a_1 d$ and $d = (a, b)$. But these remainders are exactly the numbers $0, d, 2d, \dots, (a_1 - 1)d$ because all of them are divisible by d and they are pairwise distinct. This completes the proof. □

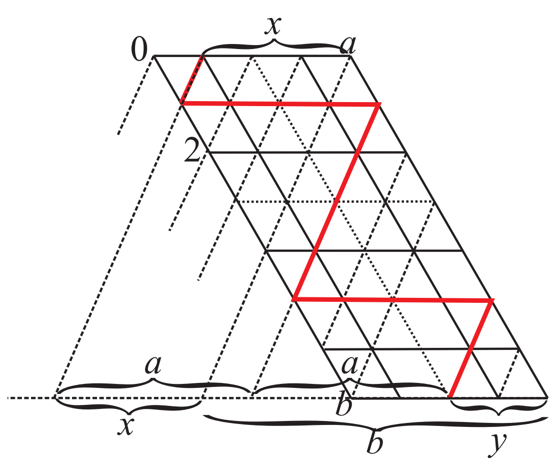


Figure 5: Relation between upper and lower impact point

Proof of theorem 2. Using the same observation as in the previous proof the coordinates of the collision points on the lower boundary segment are the remainders of $l(a - kd')$ modulo a where $k, l \in \mathbb{N}^*$. But the above remainders are exactly are exactly the multiples of $gcd(d', a)$. □

Proof of theorem 3. The assertions of theorem 3 are a direct consequence of theorem 1 and the representation of states on the billiard table. □

Remark 5. *If $c < a+b$, there are cases when not all quantities can be measured. If $a = 7$, $b = 11$ and $c = 13$, we can prove (using a Poisson type representation of all possible states) that we can't measure 1 liter.*

Proof of theorem 4. From theorem 2 and the detailed construction (see figure 4) we deduce that in the pot a_1 we can measure every quantity which is a multiple of d and does not exceed a_1. When using the table with side lengths a_j and a_{j+1} (and the corresponding pots) we consider that all the other pots a_k with $1 \leq k \leq n$ are empty and a_{n+1} contains the rest of the water. By a symmetry argument this can be extended to all pots a_j, $1 \leq j \leq n$. We can obtain all quantities in the a_{n+1} pot if for every combination of the unused pots we can repeat the same steps while this fixed combination of the unused pots is considered to be full and the rest of the unused pots are empty. □

Problem solving experience

We had been working with $10-14$ years old students and we divided them in 2 groups. In the first group there were 60 $10-12$ years old students, while the second group there were 60 $12-14$ years old students. The students were chosen randomly from 3 different Romanian cities and they were asked to solve the following exercises:

1. We have three unscaled pots with $7l$, $17l$, $24l$ volumes. Initially the largest pot was filled with water.
   a) Measure out $1l$ of water in one of the pots.
   b) Measure out $1l$ of water in the largest pot.
   c) Characterize all quantities that can be measured out in the pots.
2. We have three unscaled pots with $21l$, $34l$, $55l$ volumes. Initially the largest pot is filled with water. Measure out $1l$ of water in one of the pots.

Our problem solving activity has been designed in order to see how our students were approaching such problems. The students had to specify not only the outcome of their solution, but also their thoughts, attempts and failures as well. We have to mention that we did not solve similar exercises with the students before this activity.

The puzzling nature of the problems ensures that the students could not see the solution all at once. We expected the students to make random steps (fillings) and to realize that they must avoid the previous states. We were hoping that the students will be able to perform a sufficiently large number of steps before giving up. We suspected that there will be significant differences between the results of the two groups.

In the first group there were only a few correct solutions to exercises 1/a,b, and no solution to the exercises 1/c and 2. In the second group there were significantly more solutions to the exercises 1/a,b, a few almost correct solutions to the exercise 1/c and no solution to exercise 2. We were surprised because 60% of the first group and 45% of the second group did not understand the exercises at all. Some of the students wanted to scale the pots, others simply wanted to pour half of the water from the pot c to b and some of them wanted to pour out 1 liter measuring only with eyes. We were surprised because this kind of mathematical problems appear in many textbooks and competitions for $10-12$

years old children. From the first group the students who understood the mathematical problem were not able to perform out the necessary steps. They gave it up after the $6^{th} - 9^{th}$ correct steps and they started it over by implying false ideas, similar to their colleagues whom did not understand the mathematical problem. Probably their working memory became full and they were unable to erase it (this idea seemed to be confirmed by some comments the students made: "my brain has been blocked" or "you must measure it until you get tired"). The same phenomenon appeared in the second group as well, however the number of correct steps made toward the result was significantly higher, and about 23% of the students succeeded in solving 1/a,b.

None of these students realized that their choices (pouring from pot x to pot y) were random and they didn't try simultaneous alternative ways. Although there were no explanation on the selection of the solution, the comments of some of the students showed that they simply tried to avoid the previous states and at every state they have been choosing the next step randomly ("we just have to fill the pot till the desired quantity appears").

By comparing the histograms for the number of correct steps we obtained, it revealed a significant difference between the results of the two groups. The students from the second group were able to carry out much more steps than the students from the first group.

We also observed another interesting correlation: if we consider only those students who solved exercise 1/a and we look for a regression between the number of steps used in the first problem and the number of performed steps at the problem nr. 2, then we get two well correlated data sequence. This correlation showed that the students performed 20% less steps with the larger pots than with the smaller ones before giving up. Some of the students believed (they described it in their comments) that the second exercises can not be solved because the pots were too large. This shows that the operational skills of our $13 - 14$ students regarding addition and substraction are not yet really operational.

Concluding remarks

- The use of diagrams in solving routine or non-routine mathematical problems has been widely studied in the literature (see [5] and the references therein). The representation used by Poisson is a typical hierarchy (branching) structure (see [6]) while the billiard ball representation can be viewed as a dynamical diagram. In our case the key element of the proof is contained in the dynamical structure and it is not present in the hierarchy structure. We believe that such dynamical diagrams can be used with a greater efficiency in teaching/learning activities than the usual static diagrams. It would be interesting to develop a deeper study on the effectiveness of using dynamical representations in problem solving.

- We also wish to point out that the construction of a dynamical diagram eases the understanding of the problem. Although the original problem is a non-routine one (in our case), once the corresponding diagram has been understood, the problem becomes a routine problem.

- Our problem solving activity illustrates that in many classroom activities the miracle just happens, and the solution appears without further or deeper understanding of the phenomenons, moreover our students are familiar with this sudden appearance of a solution. The students are perfectly satisfied if they obtain a solution and they seldom search the reasons behind it. This can be a major obstacle in understanding mathematics and in developing an active and conscious attitude in doing mathematics.

- Our students did not balanced their possible choices and what is even worst most of them did not realized that they have choices and that they can experiment the effect of these choices.

- Our computer simulations show that the solution of both problems can be obtained by random steps if we avoid the previous states (and even if we do not avoid cycles, but the number of steps in this case is much more greater), so the failure of our students can not be explained neither by the defective knowledge nor by the absence of their talent or combinatorial skills. They did not have sufficient perseverance to perform as much steps as it was needed. We hope that by understanding the nature of this problem and the source of their failure our students realized what Jim Watson says about persistence: "A river cuts through rock, not because of its power, but because of its persistence."

Acknowledgements

Our teaching activity was part of the project DQME II[4]. We are very grateful to our colleagues Hajnalka Csapó (Márton Áron Highschool, Miercurea Ciuc), Jutka Szilágyi (Báthory István Highschool, Cluj Napoca), Emőke Szilágyi and István Mátéfi(Bolyai Farkas Highschool, Târgu Mureş) for their help in designing the teaching activities. The first author's research was partially supported by the Hungarian University Federation of Cluj Napoca.

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