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Intuiting mathematics from computer visualizations*

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Abstract

In this paper we investigate some possible applications of the computers in teaching mathematics. Our main goal is to illustrate how mathematical intuition and conceptualization can be supported by the computer. In the first example we use a simple mathematical model of a single product market in order to formulate and to study the stability of a fixed point for a real function $f : [a, b] \to [a, b]$. In the second part we use a simplistic mathematical model of the labor market in order to study the properties of stochastic matrices. Both examples were used in an introductory course of dynamical systems for computer science students for several years.

Keywords: visualization, conceptualization, mathematical modelling, Perron-Frobenius theorem, stability of fixed points

MSC: 97U70, 97R20, 97I20

1. Introduction

The visualization of mathematical objects and phenomenons in the computer era became a very powerful tool in teaching mathematics and it is intensively studied from several viewpoints (see [7], [3], [8], [2], [4], [9]). In this paper we illustrate how mathematical intuition, conceptualization and abstractization can be supported by computer visualization. In our examples the mathematical models and the computer visualizations are used as a foundation for an inquiry based approach and the main target is the abstractization process. We used these examples with upper secondary students (17-18 years old) and computer science students (19-20 years old) in the framework of an introductory course to dynamical systems. The main activities were organized as lab experiments (on computers) and the role of the course was just to summarize and structure the observations made by the students. The practice shows that students are able to rediscover and formulate mathematical theorems (such as the Perron-Frobenius theorem) on their own.

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2. Examples

Problem 2.1 (Single product market). Study the demand and the supply level for a single product if we assume the following

- there is a maximum level k_d for the demand;
- the difference between the maximum level and the actual level is directly proportional to the actual price;
- there is a minimal level k_s for supply;
- the difference between the actual supply and the minimal supply is directly proportional with the former price.

If we denote by $D(n)$ the demand level in the n^{th} period, by $S(n)$ the corresponding supply level and by $p(n)$ the price, then due to our assumptions we have

$$D(n) = -c_d p(n) + k_d \tag{2.1}$$

$$S(n+1) = c_s p(n) + k_s, \tag{2.2}$$

hence in the equilibrium state (when the supply equals the demand) we have $S(n) = D(n)$, for $n \geqslant 1$. This implies

$$p(n+1) = A \cdot p(n) + B, \quad n \geqslant 1,$$

where $A = -\frac{c_s}{c_d}$ és $B = \frac{k_d - k_s}{c_d}$.

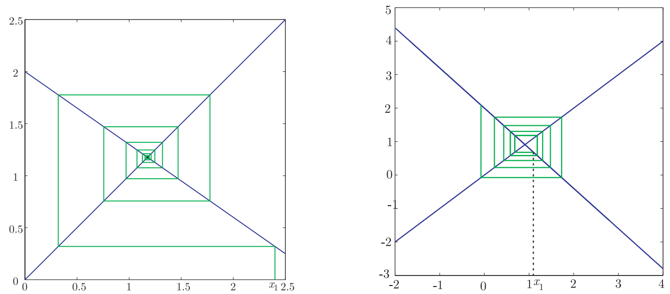

Figure 1: Stability and instability of the price

We are interested in the following questions:

- Is there any stable state, when the price is constant?
- What kind of typical behaviors can have such a system on a long time period? Is it predictable?

- What kind of typical behaviors can appear in general when $p(n+1) = f(p(n))$, where $f : [a, b] \to [a, b]$ is a real function? What are the properties of f which guarantees these behaviors?

By using the cobweb method to visualize the terms of the sequence $(p(n))_{n\geqslant 0}$ we obtain three typical cases: asymptotic stability (the sequence $p(n)$ converges to the fixed point), instability (the fixed point is a repelling one) and periodicity (for each initial value $p(1)$, the sequence $p(n)$ is periodic).

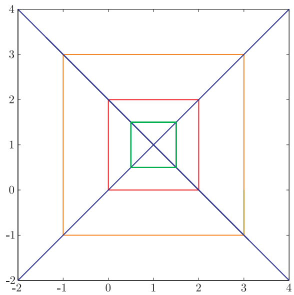

Figure 2: Stable but not asymptotically stable orbits

With a little experimentation (using a Matlab GUI, or a Flash animation) the students can explore these typical behaviors and they realize that these can be characterized by the slope of the line $y = Ax + B$. This is very helpful in treating the general case. Using a computer simulation it is obvious to realize that in general we need the tangent to the graph of the function f in the fixed point (x^*) and the behavior near this fixed point is determined by the slope of this tangent line, namely $f'(x^*)$. More precisely if $|f'(x^*)| < 1$, the sequence $(p(n))_{n\geqslant 1}$ converges to the fixed point x^* while if $|f'(x^*)| > 1$ the fixed point is repelling. In both cases the fixed point is called hyperbolic.

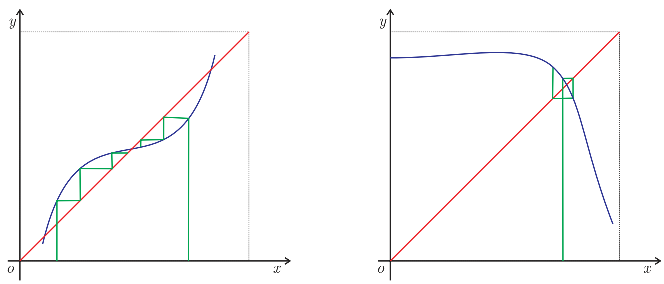

Figure 3: Asymptotically stable and unstable hyperbolic fixed points

The nonhyperbolic cases can be explored separately. If $f'(x^*) = 1$ and $f''(x^*) \neq 0$ we can observe that the fixed point behaves like an unstable fixed point at one side and at the other side it behaves like an asymptotically stable fixed point, this motivates the definition of the semistable fixed point and shows that it is necessary to have $f''(x^*) = 0$ to have stability or instability.

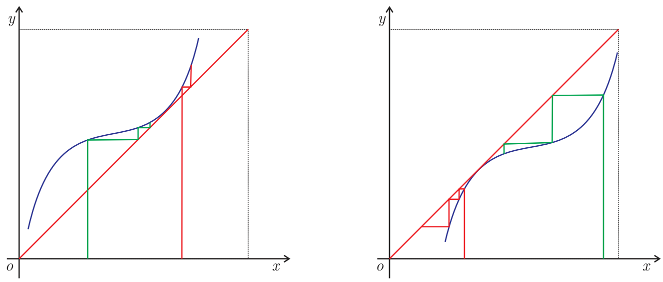

Figure 4: Semistable fixed points

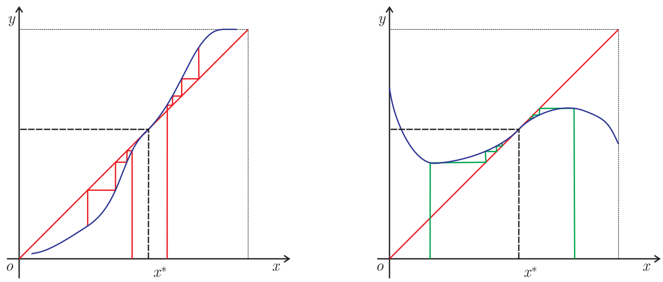

Figure 5: Unstable and stable fixed points in the case $f'(x^*) = 1$ and $f''(x^*) = 0$

By examining the graphs in this case we can remark that asymptotic stability occurs when f'' decreases near x^* and instability occurs when f'' increases. But locally these facts are equivalent with $f'''(x^*) < 0$ respectively $f'''(x^*) > 0$. This shows that the students can formulate a first theorem about the characterization of the fixed points. Moreover it is absolutely clear, that in the proof of this theorem we need a local representation of the function f using the values of its derivatives in x^*. Hence the main tool in the proof is the Taylor expansion around x^*. Based on the above reasoning we have the following theorems (see [6],[5],[10],[1]):

Theorem 2.2. *If x^* is a fixed point of the function $f : \mathbb{R} \to \mathbb{R}$ and f is continuously differentiable in the neighborhood of x^*, then we have the following properties:*

a) if $|f'(x^*)| < 1$, then x^* is an asymptotically stable fixed point;

b) if $|f'(x^*)| > 1$, then x^* is unstable.

Theorem 2.3. *If x^* is a fixed point of the function $f : \mathbb{R} \to \mathbb{R}$, $f'(x) = 1$ and f''' is a continuous function in a neighborhood of x^* then we have the following properties:*

a) *If $f''(x^*) \neq 0$, then x^* is unstable (semistable).*

b) *If $f''(x^*) = 0$ and $f'''(x^*) > 0$, then x^* is unstable.*

c) *If $f''(x^*) = 0$ and $f'''(x^*) < 0$, then x^* is asymptotically stable.*

The formal proofs can be found in [6], [1] and the key steps are in fact the formal descriptions of the observed phenomenons. By analyzing the proofs too, we can observe that almost the same argument can be applied to prove also the general version of the previous theorems, which was published in [5].

In the above cases the convergent sequences were always monotonic. If we analyze some examples where $f'(x^*) = -1$, we can observe that the sequences can converge without being monotonic, but in any cases we can split these sequences into two monotonic subsequences. More precisely in this case we need to study the function $g = f \circ f$. By applying this idea and the previous theorems we can easily obtain the following theorem from [5] and with some computations we can also obtain the main theorem of [10].

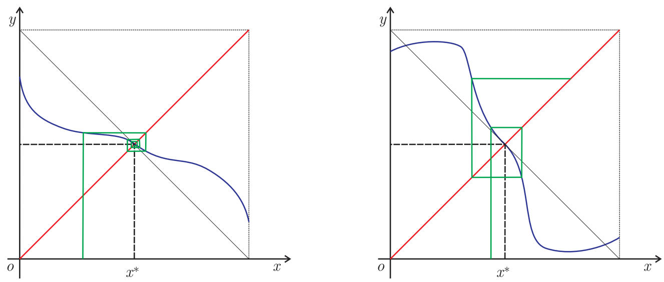

Figure 6: Stability and instability when $f'(x^*) = -1$

Theorem 2.4 ([5]). *If x^* is a fixed point of the function $f : \mathbb{R} \to \mathbb{R}$, f is three time continuously differentiable in a neighbourhood of x^*, $f'(x^*) = -1$, then we have the following properties*

a) *If $-f'''(x^*) - \frac{3}{2}(f''(x^*))^2 < 0$, then x^* is asymptotically stable.*

b) *If $-f'''(x^*) - \frac{3}{2}(f''(x^*))^2 > 0$, then x^* is unstable.*

Problem 2.5 (Labor market model). Study the population of a region under the following assumptions:

- the population is divided into three subcategories:
  - those whose job is related to their qualification;
  - those whose job is not related to their qualification;
  - unemployed;

- in each year the s_{ij} proportion of the i^{th} category moves to the j^{th} category.

Denote by a_n, b_n and c_n the number of individuals in the three aforementioned categories at the end of the n^{th} year. From the given assumptions we can obtain the following equations:

$$\begin{aligned} a_{n+1} =& s_{11}a_n + s_{21}b_n + s_{31}c_n \\ b_{n+1} =& s_{12}a_n + s_{22}b_n + s_{32}c_n \\ c_{n+1} =& s_{13}a_n + s_{23}b_n + s_{33}c_n. \end{aligned}$$

If we use the matrix

$$S = \begin{bmatrix} s_{11} & s_{21} & s_{31} \\ s_{12} & s_{22} & s_{32} \\ s_{13} & s_{23} & s_{33} \end{bmatrix} \tag{2.3}$$

and the vectors $u_n = [a_n \ b_n \ c_n]^t$, $n \geqslant 1$ the recurrence can be written in the form

$$u_{n+1} = S \cdot u_n.$$

From this relation we obtain $u_{n+1} = S^n \cdot u_1$, hence the long term behavior of the population depends on the exponents of S.

A few natural questions regarding the evolution of this population:

- For a fixed S is there any equilibrium population (in which the size of the categories remains constant)?
- Is there any structurally stable population (in which the relative size of the categories remains constant)?
- Is there any pattern in the long term evolution of this population?
- Is there any possibility to reduce the size of the unemployment category?

Figure 7 illustrates the relative sizes of the categories for $1, 5, 10, 15, 20, 25$. Using a simulation which calculates and illustrates these terms for randomly generated initial values we can see that the sequence converges and the convergence is very fast. Moreover the limit of the sequence $(u_n)_{n\geqslant 0}$ is a solution of the equation $u = S \cdot u$. This shows, that 1 is an eigenvalue for S, the corresponding eigenvector has positive components and all the other eigenvalues satisfy the inequality $|\lambda| < 1$. This is in fact the Perron-Frobenius theorem for stochastic matrices:

Theorem 2.6. *If $S \in \mathcal{M}_n(\mathbb{R})$ has positive elements and the sum of elements in each column is* 1, *then*

- 1 *is a simple eigenvalue of S and for any other eigenvalue λ the inequality $|\lambda| < 1$ holds;*
- *the components of the eigenvector corresponding to the eigenvalue* 1 *have the same sign;*
- *the sequence $S^n \cdot u_0$ converges to u^*, where u^* is an eigenvector corresponding to the eigenvalue* 1 *and the sum of the components in u^* is the same as in u_0.*

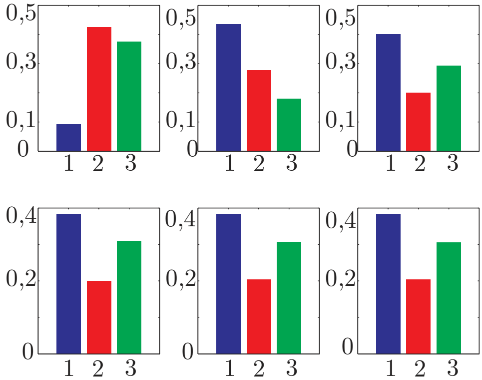

Figure 7: Convergence of iterates for a stochastic matrix

3. Concluding remarks

- In both examples we used the model to support the intuition, the understanding and the formalization of mathematical theorems. Although the theorems are not included in the regular curricula (for upper secondary school), these examples can be used in order to help students in understanding and deepening the corresponding mathematical notions. In many countries the mathematics curriculum and the final examination criteria does not include modelling skills, they focus more on problem solving and abstract mathematical notions, theorems. In such a framework the modelling activities can (and must) be used to facilitate the understanding of mathematical phenomenons.

- The main importance of this approach is that the students can formulate abstract theorems based on their own experiments, they can understand the key steps of the proofs based on computer visualization. In performing this inquiry based approach the use of a computer is of crucial importance. In the first example most of the students can formulate theorems 2.2 and 2.3 while in the second example most students can formulate the whole theorem. Without computer this is not so palpable (especially in the second case), because the volume of the calculation is too big.

- Both examples show that if we use an inquiry based approach (specially in secondary school and upper secondary school) from very natural questions we can arrive to deep mathematics (remember what Dean Schlicter said: Go down deep enough into anything and you will find mathematics) which formally exceeds the existing curricula. If we want to train our students in using/maipulating/creating high complexity reasoning, then the curricula should be more flexible in terms of the content and much more focused on competencies, processes.

- The use of computer visualizations can attract a wider audience, but in order to prepare them the teaching staff needs a special training (in creating flash animations, interactive graphical user interfaces), hence the teacher training curricula must include special topics on realizing visualizations, animations.

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[1] Developing Quality in Mathematics Education, for more details see http://www.dqme2.eu/