# Constructing with nonstandard bricks

Szilárd András[1]
Babeş-Bolyai University, Cluj Napoca, Romania

Play is our brain's favorite way of learning.
*Diane Ackerman*

Abstract. In this short note we present a student led inquiry-based activity built upon playing with nonstandard toy construction bricks. The main aim of the activity was to create an environment in which well contoured mathematical content (counting, linear diophantine equation, system of linear equations, inductive arguments, unfolding of a cube) can be emphasized along students questions. We used this activity with lower and upper secondary school students and also with a group of teachers in a professional development course, with a slightly different focus depending on the participants.

Keywords: mathematical modeling, student led inquiry
Mathematical Subject Classification: 97M99, 97A20

## Introduction

The necessity of using inquiry-based learning (IBL) was recently recommended by scientific studies and reports made for the European Commission (see [4]). Several European projects are devoted to the widespread of IBL methods (see the ProCoNet group at http://proconet.ph-freiburg.de/). Moreover, the effects of using IBL are studied worldwide (see [6]). In the framework of the FP7 project PRIMAS[2] a series of piloting activities were organized in Romania in order to test, adapt and develop inquiry-based teaching materials. Most of these piloting actions were organized by local professional communities with the purpose of creating a real feedback for the project and for gathering professional experience in implementing inquiry-based pedagogies in mathematics and science education. The main aim of this paper is to present an activity where students were formulating the problems, teachers were only creating the milieu ([3]) and facilitating the work. As a second step the accumulated experience related to this activity was used in a PD course organized by the Babeş-Bolyai University in the framework of the PRIMAS project.

During the activities students were working in small groups, each group had two types of pieces shown in Figure 1 and they had to construct and plan patterns

[1]Email address: andraszk@yahoo.com

[2]Promoting inquiry in Mathematics and science education across Europe, Grant Agreement No. 244380
http://www.primas-project.eu

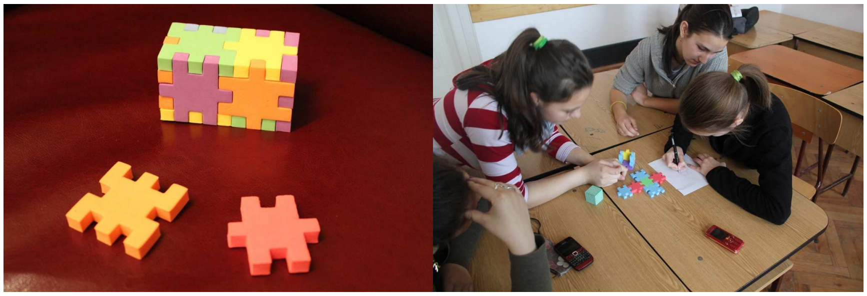


Figure 1: The pieces and the $5 \times 5 \times 5$ cube unfolded

and $3D$ objects. The patterns and objects were not specified, each group decided what to construct. One basic rule was fixed: all groups had to present all successful and unsuccessful attempts, moreover for all the constructions it was necessary a careful analysis. It is worth to mention that the number of pieces was relatively small for each group (12-36), but in constructing the patterns and objects they had to deal with an unlimited number of pieces.

## Constructing patterns and 3D objects

Before our activities we organized a series of playing activities with the set of Marble and Profi cubes (see http://www.happycube.com/ and [1]). The first $3D$ object constructed was the cube, while in the plane most of the groups constructed rectangular configurations. After these they constructed rectangular parallelepipeds (see Figure 1), square pipes, curved square pipes and different other shapes (see Figure 2).

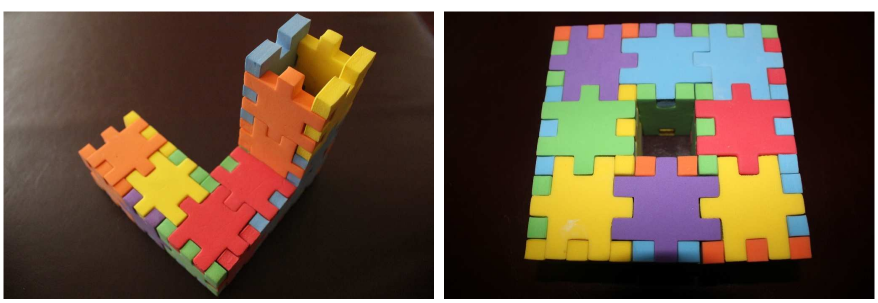


Figure 2: Curved square pipe

During the analysis of the plane configurations most groups observed that if we want to cover a certain planar region, we don't have too much options. In fact there exists only one possible way to fit the pieces and this arrangement generates a covering of the plane (see Figure 5). The formal proof of this fact was not requested, but the students formulated an inductive argument, which emphasized that a configuration formed by 4 pieces is repeated.

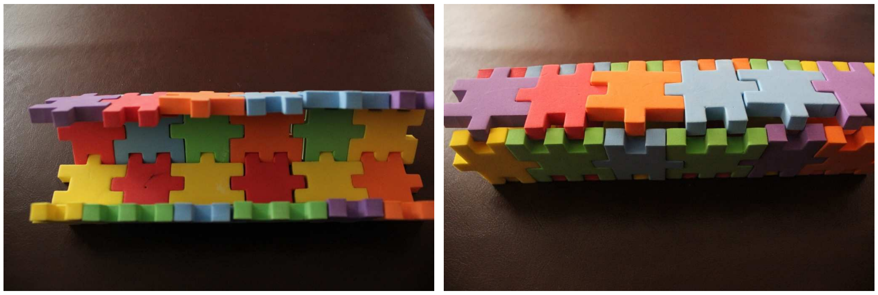


Figure 3: Constructing a square pipe from the plane covering

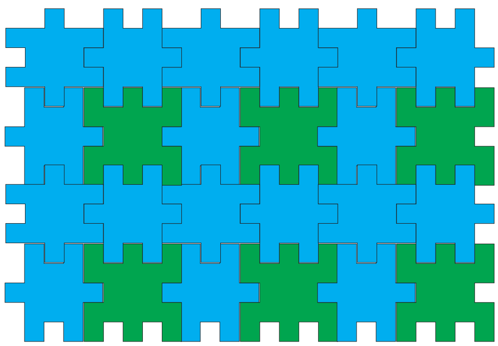


Figure 4: Unfolded square pipe

Similar arguments were given for the construction of square pipes, moreover, students found a connection between the covering of the plane and the construction of the pipe. They observed that if we unfold the pipe, we obtain a strip from the covering of the plane. This observation assures that a pipe with arbitrary length can be constructed (see Figure 4 and Figure 3).

Using an additional argument students showed that any curved pipe can be constructed (the ratios of straight sections can not be arbitrary, but the direction can be changed at any step, see Figure 2). The determination of the number of needed pieces for such a construction was also carried out by the students.

The first surprise appeared when students wanted to construct a $9 \times 9 \times 9$ cube (which should contain $2 \times 2$ pieces on each side) and they failed.

Atfer a few attempts they formulated that it is impossible to construct a $9 \times 9 \times 9$ cube (without holes). Some groups constructed a $9 \times 9 \times 9$ cube where 2 small $(1 \times 1 \times 1)$ cubes from the opposite corners were missing, so they also conjectured

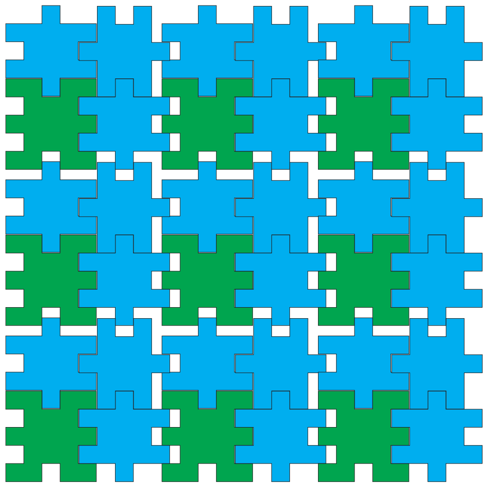


Figure 5: Covering of the plane

Figure 6: Unsuccesful attempts to construct a $9 \times 9 \times 9$ cube

that this is the best approximation of the $9 \times 9 \times 9$ cube with the given pieces. In order to prove these statements student constructed the following proofs.

**Proof 1.** Count the total numbers of $1 \times 1 \times 1$ cubes on the surface of the $9 \times 9 \times 9$ cube. If it is possible to construct a $9 \times 9 \times 9$ cube from the given pieces, than the pieces will fill/cover exactly these $1 \times 1 \times 1$ cubes. So if we use $x$ pieces of type $I$ and $y$ pieces of type $II$ (see Figure 7), then $15x + 19y$ is the total number of $1 \times 1 \times 1$ cubes on the surface (including those in the corners and on the edges of the cube). On the other hand the total number of the small cubes on the surface is 386, so we obtain an equation for $x$ and $y$.

$$15x + 19y = 386 \tag{1}$$

**Remark 1.** *With lower secondary school students the problem of counting the unit cubes on the surface was not a trivial one, hence we needed to discuss several counting techniques. Some groups decomposed the cube into disjoint parts, an upper and a bottom* $9 \times 9 \times 1$ *part and the rest, with height* 7 *and circumference composed by* 32 *units. Other groups used a logical sieve technique and there were also some other decompositions.*

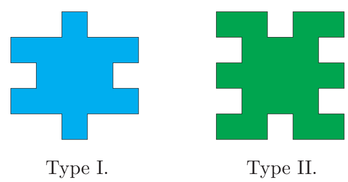



Figure 7: The pieces of type I and II with 15 and 19 small cubes respectively

Some groups considered equation (1) as a diophantine equation, determined all nonnegative solutions and proved that these solutions are not realizable. The only solution is $(8, 14)$ and this would imply that there is at least one face having 3 pieces of type II, which is impossible.

Other groups observed that the number of pieces is also known, because each face is composed by 4 pieces, so there are 24 pieces. This can be expressed as $x + y = 24$, so together with (1) we obtain a system of equations, which has no integer solutions. This implies that there is no $9 \times 9 \times 9$ cube. In order to prove the conjecture for the best approximation we need to solve also the system

$$\begin{cases} 15x + 19y = 385 \\ x + y = 24 \end{cases}$$

The solutions of this system are not integers, so there is no configuration which covers all the surface with one exception. This implies that the conjecture about the "best" approximation is correct. Of course this result raises the question of finding all such configurations. We will obtain answer to this problem from Proof 4.

**Proof 2.** Some groups observed that a face can be constructed only in one way (only the configuration generating the plane covering can appear). Starting from this they tried to combine these faces on a planar skeleton in order to obtain the unfolded image of the cube. This matching of the unfold faces can be done in a unique manner, so they had to verify only if their diagram is a real unfolded cube. They found quickly a contradiction on the diagram, so they obtained a proof (see Figure 8).

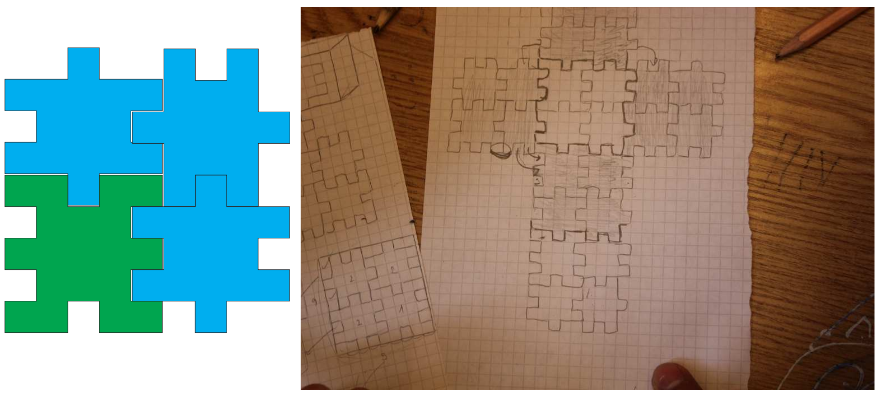


Figure 8: The only possibility for the face of a $9 \times 9 \times 9$ cube and a proof of impossibility

**Proof 3.** Other groups counted only the unit cubes on the edges and at the vertices of the $9 \times 9 \times 9$ cube. The number of these cubes is 92, hence the 6 faces has to cover these cubes. But the faces are congruent (because there is only one possible way of combining 4 pieces into a face), so 92 should be divisible by 6. This contradiction shows that the cube can not be constructed. Moreover, each face covers 15 unit cubes on the edges and vertices, so 2 unit cubes will remain uncovered and these are on the edges or at the vertices.

**Proof 4.** One group observed that the cube has 8 vertices, while each face (as shown in Figure 8) can cover only one, so 2 vertices can not be covered.

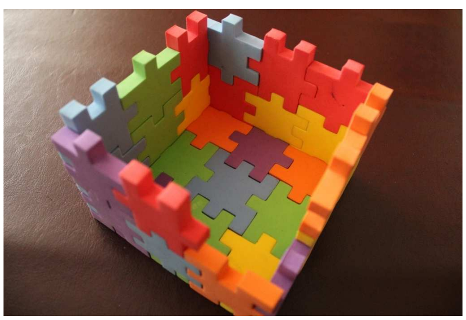
The above problem raised a general question: which cubes can be constructed from these pieces. Or more generally characterize all the parallelepipeds that can be constructed from the given pieces.

At the activities before attacking this general problem we first discussed all the previous ideas and each group had chosen one more special case to analyze. Moreover we changed the notations in order to handle the number of pieces. In what follows we say that a cube is $1 \times 2 \times 3$ if faces contains $1 \times 2$, $1 \times 3$ and $2 \times 3$ pieces. Hence the real dimensions (in units as before) are $5 \times 9 \times 13$. In general in what follows if we say $k_1 \times k_2 \times k_3$ (in pieces), this means $(4k_1+1) \times (4k_2+1) \times (4k_3+1)$

in units. The following particular cases were analyzed: $1 \times 2 \times 3$, $2 \times 2 \times 1$, $3 \times 3 \times 3$, $4 \times 2 \times 1$, $4 \times 4 \times 4$, $5 \times 5 \times 5$. Using the previous ideas (or combinations of them) the following results were established (including those studied before):

- the parallelepipeds $1 \times 1 \times 1$, $1 \times 2 \times 3$, $3 \times 3 \times 3$, $5 \times 5 \times 5$ can be constructed by using the given pieces;
- the parallelepipeds $2 \times 2 \times 1$, $4 \times 2 \times 1$, $4 \times 4 \times 4$ can not be constructed by using the given pieces.

Figure 9 shows a solution for the cube $3 \times 3 \times 3$. Similar figures were drawn for all possible constructions. For all the previous impossible cases a proof was detailed. Based on these cases the groups formulated the following general property:



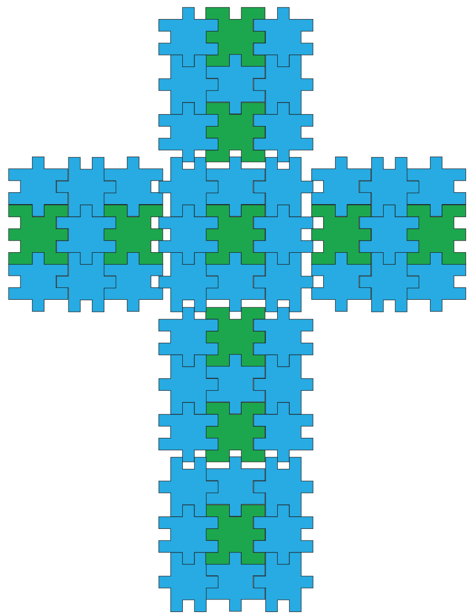


Figure 9: The construction of a $13 \times 13 \times 13$ cube and it's unfolded skeleton

**Theorem 1.** *The parallelepiped $k_1 \times k_2 \times k_3$ can be constructed if and only if at most one of the numbers $k_1, k_2, k_3$ is even.*

The proof of this property consists of two steps. In the first step we prove that if there are at least 2 even numbers among $k_1, k_2, k_3$, then the parallelepiped can not be constructed from the given pieces. This can be done using several different approaches. The simplest is to use the counting argument and the system of linear equations. The number of pieces is $2(k_1k_2 + k_2k_3 + k_3k_1)$, while the number of unit cubes on the surface is $32(k_1k_2 + k_2k_3 + k_3k_1) + 2$, hence we obtain the following system:

$$\begin{cases} 15x + 19y = 32(k_1k_2 + k_2k_3 + k_3k_1) + 2 \\ x + y = 2(k_1k_2 + k_2k_3 + k_3k_1) \end{cases}$$

From this system we obtain $2y = k_1k_2+k_2k_3+k_3k_1+1$, so if $k_1k_2+k_2k_3+k_3k_1$ is even, the solutions of the system are not integers. On the other hand if there are at least 2 even numbers among $k_1, k_2, k_3$, then $k_1k_2 + k_2k_3 + k_3k_1$ is even. Unfortunately the fact itself that the system has integer solutions doesn't imply that the corresponding parallelepiped can be constructed, so we need the second step for proving that the parallelepiped can be constructed if at most one of the numbers $k_1, k_2, k_3$ is even. Suppose $k_1$ and $k_2$ are odd. As in Figure 9 we can construct the skeleton for the unfolded version by considering the column with the alternating type I and type II pieces in the middle and the face common to the main row and column of the diagram being $k_1 \times k_2$. Due to the periodicity of the covering only the cases $k_3 \in \{1, 2\}$ need to be analyzed and this was done previously (at the particular cases).

**Remark 2.** *At the activities the last problem was treated only with uppers secondary school (16-17 years old) students and teachers. At some activities with lower secondary school (14-15 years old) students we had to finish with the* $13 \times 13 \times 13$ *cube. With teachers we could discuss all the presented aspects in* 4 *hours, while working with students we usually had* 3 *sessions,* 2 *hours long each.*

# Final conclusions

1. At such an activity the greatest problem is how to convince students to stop playing and to start doing mathematics. This problem was controlled successfully by the small number of pieces the students had. None of the groups had enough pieces to construct a $9 \times 9 \times 9$ (in units) cube, so they were forced by the milieu to focus on the mathematization, on the modeling of the problem.

2. We used this activity with several different groups and several different groups of teachers. The final theorem was not formulated all the time (not even with teachers), because other important aspects needed to be clarified (such as the sieve method for counting, or the properties of the linear diophantine equation, including the existence of positive solutions), but all the activities had very substantial mathematical content, and this content was developed along the questions posed (or difficulties faced) by the participants.

3. The teachers participating at the professional development course were very surprised about the number of correct answers/arguments that can be given to a specific problem. They understood the importance of analyzing the ideas of students (even if at first sight they seem not effective for some reasons).

4. The pieces we used are from a simplified set of Happy Cube type puzzle set (for an analysis of the original set see [1]), but similar activities can be designed also with other sets of pieces. In fact we used a trivial covering of the plane (the covering with squares) to generate a lot of interesting questions. This can be done starting from other coverings too (triangular covering, hexagonal covering, pseudo regular coverings, etc.).

5. From the perspective of the learning processes and students/teachers attitudes and interactions these activities were very instructive both for students and teachers. All presented ideas, solutions, arguments, counterarguments, examples were formulated by participants. The teachers running the activities were only facilitating the discussions inside the groups and among groups. In this way students had the occasion to understand (on a small scale) how mathematics and mathematicians work. They also had the occasion to compare different approaches (they developed) for the same problem and to highlight the advantage/disadvantage of each approach. On the other hand teachers at the PD course realized the importance of the facilitation process, where the major focus is on what the participants can realize and not the apriori, content related knowledge of the teacher.

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