Beads and formulas

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Inquiry is fatal to certainty.
Will Durant

Abstract. The main aim of this paper is to sketch an inquiry based approach to teaching Newtons's binomial theorem and the related combinatorial notions (binomial coefficients, combinations, permutations with repetitions, multinomial coefficients, Pascal triangle, Pascal tetrahedron) and to present a few classroom observations from implementing this approach.

Keywords: Pascal triangle, binomial theorem, inquiry based learning

Mathematical Subject Classification: 97K20, 97M10

Introduction

Recently the inquiry based approach in teaching mathematics and science has been acquired a lot of interest and attention in most of the European countries. This is a consequence of scientific research ([4]) and high level political recommendations, decisions ([1]). However there are considerable efforts at all levels, a lot of European Projects ([3]) are focusing on inquiry based learning in mathematics and science, the changes in day-to-day practice are not yet visible on a large scale. Moreover many mathematics and science teachers are against using IBL in their national context, because they perceive incompatibilities between the existing framework (curricula, assessment, attitudes, etc.) and the IBL approach. Our piloting actions, workshops, demonstration teachings organized in Romania by the Simplex Association and performed by the Romanian PRIMAS team shows that IBL is highly suitable for teaching most of the existing curricula if we use a more flexible time management. The main aim of this paper is to outline a possible approach in teaching Newton's binomial theorem and the combinatorial notions needed. Moreover this approach is appropriate to understand even more general relations such as the Pascal tetrahedron or the multinomial theorem. In fact this aspect (that we can use the same framework to develop deeper content) is very common for IBL teaching materials and this can be a major advantage even in working with gifted students. We used the presented approach with 15-16 years old students in the "Márton Áron" Highschool in Miercurea Ciuc and with several groups of students in summer camps. Moreover we used the same activities with pre-service teachers and with regular high-school teachers at the Primas Professional Development courses.

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The basic context

The main problem. In a village there are a lot of crones and one of them hears a rumor. Due to her uncontrollable urge she starts spreading the rumor in the next day by enlarging the rumor a times and (separately) b times and telling these versions to two other crones (who has not heard any version of this rumor). All crones in the village who hear versions of the rumor will do the same. The main task is to organize clubs in this village such that in each club we invite those crones who hear the rumor in the same enlargement (so we have to invite to the same club those who heard after enlargement with a, a and b or a, b a or b, a, a, but no other crone will be in this club). What happens if each crones tell 3 (or more) different versions to 3 other crones? □

Remark 1. *The problem can be formulated by the dilution of a chemical agent, or by selling stocks between banks.*

Remark 2. *Depending on the students we are working with, we can use more structured tasks:*

1. *prepare an illustration to represent crones and then the process of organizing the clubs;*
2. *calculate the number of crones invited to a specific club;*
3. *calculate the amount of rumor in a club;*
4. *how many clubs can be organized in the n^{th} day, how many crones will be invited to these clubs;*
5. *calculate the total amount of rumor in the clubs at the n^{th} day.*

But the best is to let students pose questions and develop their model. This phase can be organized as a first task, where students are working in groups and all the groups have to make a list of questions they want to answer. Here the teacher has the opportunity to help students focusing on questions that are relevant in order to construct a useful model.

Figure 1: Students' first attempts

Remark 3. *The main context can be modified in order to lead the students towards understanding the expansion of arbitrary algebraic products, such as*

$$(p_1 + xq_1) \cdot (p_2 + xq_2) \cdot \ldots \cdot (p_n + xq_n),$$

hence it can be used also in order to connect the algebraic calculations to probabilistic models.

The activities

At our activities we had basically the following steps:

- (5'-10') introduction to the context, the main problem;
- (10'-15') a first brainstorming to outline a series of possible questions, problems and to fix the role of groups (each group could inquire a separate set of aspects);
- (30'-40') organizing the clubs (and of course finding as much information as possible about the number of members, the amount of rumor in a club, the daily amount of rumor) for the first $2-3$ days;
- (40'-50') prepare a hand made model using a board, some nails, string and plastic (or wood) beads;
- (40'-50') fixing the ideas (introducing the Pascal triangle and it's basic properties, the expansion for $(a+b)^n$, permutations with repetitions, the binomial coefficients, the combinations);
- (additional step - 2 hours) prepare a three dimensional model for the case when each crone is spreading the rumor to three more crones.

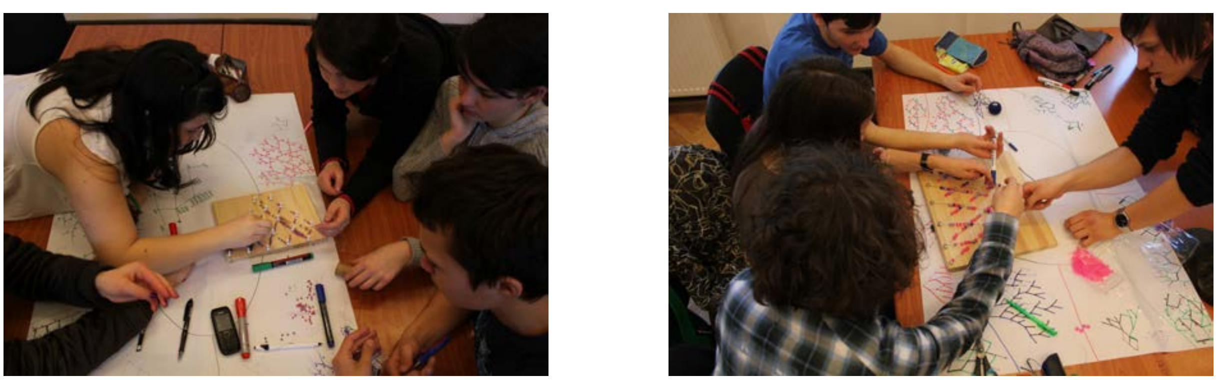

Figure 2: Preparing the $2D$ models: adding the codes to visualize combinations

Remark 4. *Working with preservice teachers or teachers, the main focus is not only the development of scientific content but also analyzing all the possible questions that can arise during these activities (both from students and from teachers), common mistakes that can appear during the activities. The most common mistake usually appears during* *the preparations of the hand made model (this shows that usually the concepts are not clear at this stage) and consists of using only one string between the nails. This can be avoided if students make a model of the binary three from strings and they try to regroup the vertices of this three according to the clubs. In this way they will realize that there are multiple strings between clubs, but for this we have to give up the use of wood board at the beginning (so the time needed will increase).*

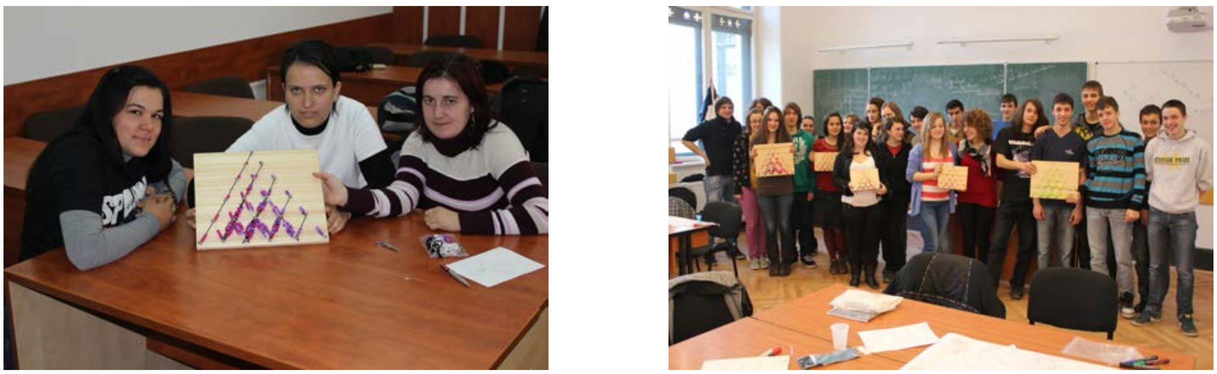

Figure 3: The $2D$ models showing not only the binomial coefficients but the combinations too

The wood boards and the nails can be used after the students realized how to form the clubs and they have some kind of representation. These tools are used to obtain a more transparent output, the same structure can be also represented on a poster as a figure.

If we need to gain some time we can present a modified context, where the wood board and the nails are already fixed and they are representing crones, but the crones can hear more rumors and they are spreading all the rumors according to the same rule (enlarging them by a and b). In this case the arrangement of the nails on the board represents somehow the relations between crones (neighbors).

Figure 4: Making a $3D$ model without a rigid skeleton and preparing a skeleton

The same logic applies to the $3D$ models. In the first step it is important to understand the structure we are building, but afterwards it is extremely helpful if we have a rigid skeleton, because it gives the opportunity to focus on the important issues. Without this we need at least 5 persons in each team and for some of them the whole construction will be boring (because they will only holding the strings). Working in $3D$ it could be helpful to construct the structure of the whole model by stretching the strings to visualize the interior points corresponding to clubs. For this it is recommended to construct the model up to the 5^{th} day (level), where the expansion of $(a+b+c)^4$ appears. When we work with students in $2D$ in order to avoid early (and sometimes wrong) conceptualization it is also recommended to construct the model at least up to the 6^{th} day (level), where the expansion of $(a+b)^5$ appears.

Remark 5. *The activity for preparing the hand made model can be used independently in a traditional school setting if we give students the task of designing it. In this case they have to count the number of beads, the length of string, the number of nails that are necessary for a group to manufacture the model. This is a simple but instructive task for students in $2D$ and a little bit more challenging in $3D$.*

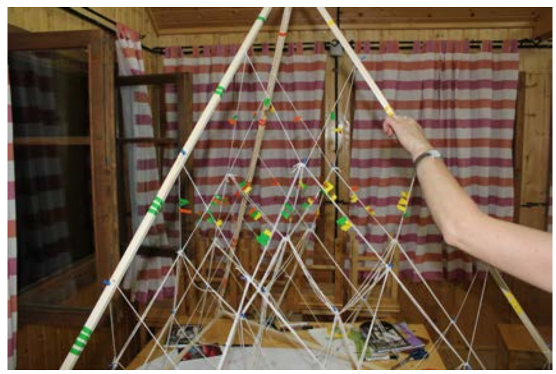

Figure 5: Making a $3D$ model and coding all the string parts

Content related remarks

Figure 6 illustrates the binary three which shows how the rumor is spread. On this three each vertex represent a crone and each edge a possible version of the rumor. This three can be rearranged in order to organize the clubs into the three on Figure 7. On this three each vertex represents a club, the number of incoming arrows represents the number of crones invited and each arrow represents a version of the rumor. In both cases the code corresponding to an arrow shows on one hand the route of the rumor (a code *aabbaab* will correspond to the route

$$left - left - right - right - left - left - right$$

from the starting point) while on the other hand if we interpret this code algebraically as the product of symbols, then this product gives the amount of rumor on the corresponding edge (hence the code *aabbaab* means a^4b^3).

Remark 6. *In the modified context, when the initial binary tree is not constructed the meaning of the objects is slightly changed, but it is still reflecting all important mathematical notions. In this setting each vertex corresponds to a crone who is hearing several* *versions of the initial rumor. For these versions the enlargements are the same, but their order is different. In fact the versions arriving to each crone are all possible (but significantly) different orders of the enlargements (here appear the permutations with repetitions).*

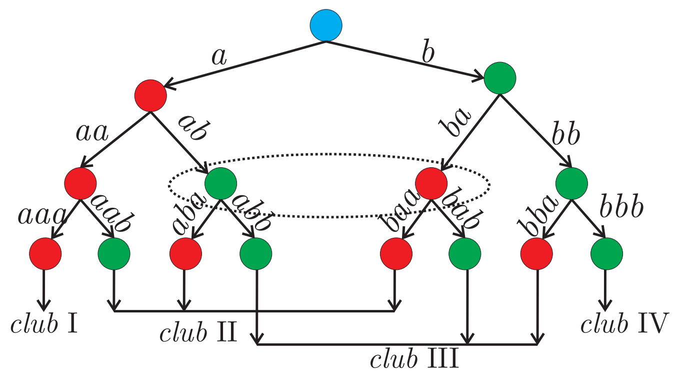

Figure 6: Rearranging the binary three on the third level

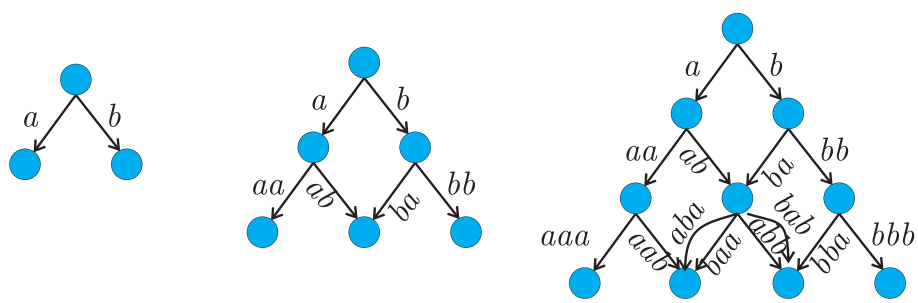

Figure 7: The clubs during the first 3 days

Remark 7. *If we consider the urns $U_1, U_2, \ldots U_n$ containing black and white balls and we extract one ball from each, than we can look for the probability of the having exactly k white balls among the extracted balls. The structure of choices for this problem (or for an equivalent real context) can be represented with the same trees and the calculation of probability reduces exactly to the rearrangement of the initial binary tree and to the determination of the corresponding coefficient in the expansion of a polynomial product, hence this activity can be used also to create the background for study this kind of problems.*

Remark 8. *The manufacturing of hand made models shows in fact an acceptable algorithm for generating the combinations on a computer. This shows that as a by-product we have obtained also a solution to a problem which usually is related to computer programming.*

In this context it is easy to observe that the total amount of rumor on the $(n+1)^{th}$ day is $(a+b)^n$, where $n \in \mathbb{N}$, the number of crones, who hear some version of the rumor on the $(n+1)^{th}$ day is 2^n and this is the total number of members in the clubs on level $(n+1)$.

The first figure illustrates the permutations with repetitions of 2 elements, where the first element is repeated once and the second element is repeated 4 times. The number of such permutations is the number of incoming strings, while the permutations appear on the strings. The second figure illustrates the permutations with repetitions of 2 elements, where the first element is repeated 2 times while the second element is repeated 3 times.

Figure 8: The number of permutations with repetitions and the permutations themselves

This shows that the sum of binomial coefficients on the $(n+1)^{th}$ row is 2^n. Moreover in a club the total amount of rumor is the product of the coefficient (the number of members) and the algebraic product obtained from the code of an incoming arrow, hence this is of the form $\binom{n}{k} a^k b^{n-k}$. As the total amount of rumor on a level is the sum of amounts from the clubs on that level we obtain the binomial theorem. If we want a more formal proof, the inductive reasoning is only a formal description of what we have seen already. If we count the number of incoming arrows at a club, it is obviously the sum of the two numbers above (from the previous level), so we obtain the generating rule for Pascal's triangle. Moreover this view allows (and in some sense prepares) to work with the Pascal triangle as a meaningful mathematical object in solving more complex problems like the following:

Problem 1. *On the sides of Pascal's triangle we replace the sequences* $1, 1, 1, \ldots, 1, \ldots$ *with the sequences* $1, 2, 3, \ldots, n, \ldots$ *and the rest of elements are calculated using the same rule as for the Pascal triangle. Find an expression of the* k^{th} *element from the* n^{th} *row!*

Solution. The solution consists of a single observation. If we add the corresponding elements of the triangle from the problem to the elements of the Pascal triangle, we obtain the triangle which can be obtained from the Pascal triangle by omitting the sides of this triangle (see Figure 9. and Figure 10.). This observation leads to the formula

$$x_{n,k} = \binom{n+2}{k+1} - \binom{n}{k}.$$

1		1		2
2 2		1 1		3 3
3 4 3	+	1 2 1	=	4 6 4
4 7 7 4		1 3 3 1		5 10 10 5
5 11 14 11 5		1 4 6 4 1		6 15 20 15 6

Figure 9: Adding the numbers on the corresponding positions

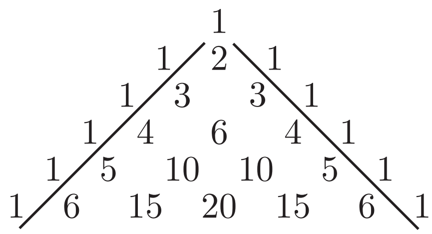

Figure 10: Omitting the sides of the Pascal triangle

□

As a last idea we give an example of a classical problem for which starting from the Pascal triangle we obtain slightly different solution than in the classical framework.

Problem 2. *Calculate the sum*

$$S_n = \binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \dots$$

Solution. If we introduce the notations $a_{n,0} = S_n$,

$$a_{n,1} = \binom{n}{1} + \binom{n}{4} + \binom{n}{7} + \dots$$

$$a_{n,2} = \binom{n}{2} + \binom{n}{5} + \binom{n}{8} + \dots,$$

we have the following system of recursions (see Figure 11. for the second relation):

$$\begin{aligned} a_{n+1,0} &= a_{n,0} + a_{n,2} \\ a_{n+1,1} &= a_{n,0} + a_{n,1} \\ a_{n+1,2} &= a_{n,1} + a_{n,2}. \end{aligned}$$

From these relations we deduce

$$a_{n,0} = a_{n+3,0} - 3a_{n+2,0} + 3a_{n+1,0} - a_{n,0},$$

which is a third order linear recurrence for the initial sequence. From this using the representation we obtains the usual formula for S_n.

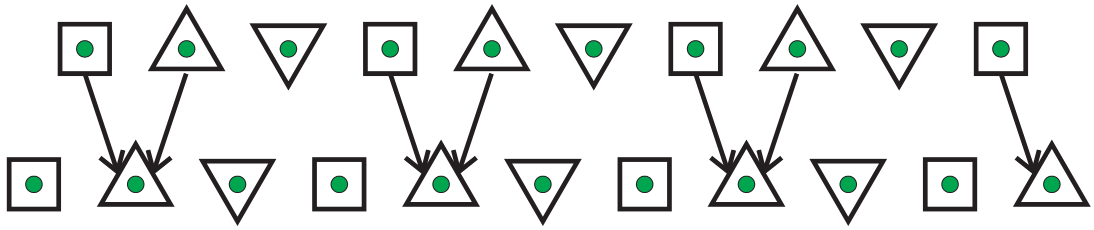

Figure 11: Recursive relations for the sequences $a_{n,1}$ (the terms of different sums are marked differently)

Remark 9. *The classical solution uses the binomial theorem and the third order complex roots of the unity, while the idea sketched above was found during an IBL activity. This is an illustrative example of how problem solving mechanisms can change if concepts are introduced by completely new activities.*

□

A major advantage of this approach is that by changing the parameters in the initial context (each crone tells the rumor to 3 other crones with the enlarging factors a, b and c), we will have a strong basis to construct the model and to understand the meaning of all the elements from the model.

Final conclusions

1. The presented activity is a good example of how a rich IBL teaching context determines the structure of the content. Hence the use of IBL methods instead of the traditional school framework (mostly based on frontal teaching) needs the rethinking of the whole structure of the content.

2. A common misconception is that IBL methods need more time. This is true only if we focus on individual activities. If we count the total teaching time necessary to teach a chapter (such as combinatorics), we can see that the longer time we spend on the introduction of concepts, on the development of properties and on the consolidation of connections between concepts is rebalanced at the practicing phase and at problem solving.

3. Remark 7 and 8 shows that if we think of knowledge construction in a longer term, than the time spent with this activity will be recuperated. Moreover if we support students in developing key knowledge structures, the total amount of time can be reduced and the time used for memorizing properties and struggling with long series of exercises in order to understand how a mathematical structure can be applied can be used for doing more interesting activities.

4. We used this activity with several different groups of (15-16 years old) students and several different groups of teachers. The activity itself is suitable for all groups, but with different age groups (which means different background level) the main focus is moving from the introduction of basic concept and identification, recognition of

basic relations to more general aspects like multinomial coefficients, combinatorial sums, teaching aspects, activity development.

5. Our approach gives a lot of opportunity to the teachers to act mostly as facilitators during most of the activities and only to introduce the well known notations for most of the objects students find during their activities, not to introduce the mathematical concepts themselves.

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