GEOMETRIC PROPERTIES OF SOME ALGEBRAIC CURVES

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Abstract. In this note we study a few classes of curves with interesting geometric properties. The definition of these classes were suggested by geometric properties of some well known curves like: straight strophoid, cissoid, the curve defined by V. Schultz and L.C. Strasznicki which contains both the cissoid and strophoid.

Definition 1.1. *The curve γ is called an $L-$type curve if admits a parametric representation of the form*

$$\begin{cases} x = \frac{f_1(t)}{g(t)} \\ y = \frac{f_2(t)}{g(t)} \end{cases} \tag{1.1}$$

where the coefficients of the functions $f_i(t) = a_i t^3 + b_i t^2 + c_i t + d_i$, $i \in \{1, 2\}$, $g(t) = a_3 t^3 + b_3 t^2 + c_3 t + d_3$ satisfy the following condition

$$b_1 = b_2 = b_3 = 0 \text{ or } c_1 = c_2 = c_3 = 0.$$

Definition 1.2. *The curve γ is called a $C-$type curve if admits a parametric representation of the form*

$$\begin{cases} x = \frac{f_1(t)}{g(t)} \\ y = \frac{f_2(t)}{g(t)} \end{cases} \tag{1.2}$$

where the functions $f_i(t) = a_i t^3 + b_i t^2 + d_i$, $i \in \{1, 2\}$, $g(t) = b_3 t^2 + d_3$ satisfy the following condition

$$\left(f_1^2 + f_2^2\right) \vdots g \text{ in } \mathbb{R}[X]. \tag{1.3}$$

Remark 1.1.
1. *It is obvious that any $C-$type curve is an $L-$type curve;*
2. *The straight cissoid defined by the equations $x(t) = \frac{rt^2}{1+t^2}$ and $y(t) = \frac{rt^3}{1+t^2}$, where $r > 0$ is a $C-$type curve (the line $x = r$ is the asymptote of the cissoid and r is the radius of the circle which generates the circle, see* [1], [9], [3]*). The curve defined by the relations $x(t) = \frac{rt^2}{t^3+1}$ and $y(t) = \frac{rt^3}{t^3+1}$ is an $L-$type curve but it is not a $C-$type curve.*

Notations. In what follows we denote an arbitrary point $M(x, y)$ on the curve by $M(m)$, which means $x(m) = \frac{f_1(m)}{g(m)}$, $y(m) = \frac{f_2(m)}{g(m)}$, if in our definitions the coefficients of t^{p-1} are equal to zero (t^p is the dominant member) and $x(m) = \frac{f_1(1/m)}{g(1/m)}$, $y(m) = \frac{f_2(1/m)}{g(1/m)}$, if the coefficients of t are equal to zero. If we use $1/m$ instead of m we consider the point at infinity as $M(0)$. In our proofs we denote the parameter of an arbitrary point X with x. Throughout this paper we suppose that the intersections in our statements do exist. We can avoid this inconvenience if we use projective plane.

The main tools in establishing our results are the following two lemmas:

Lemma 1.1. *If γ is an $L-$type curve, $A(a)$, $B(b)$ are two points on γ and the line AB intersects γ at a third point $C(c)$, then $a+b+c=0$.*

Proof. Let $\alpha x+\beta y+\gamma=0$ be the equation of the line AB. From $x(t) = \dfrac{f_1(t)}{g(t)}$ and $y(t) = \dfrac{f_2(t)}{g(t)}$, it follows that $\alpha f_1(t) + \beta f_2(t) + \gamma g(t) = 0$. But this is equivalent to $\lambda_1 t^3 + \lambda_2 t^2 + \lambda_3 t + \lambda_4 = 0$, where $\lambda_1 = \alpha a_1 + \beta a_2 + \gamma a_3$, $\lambda_2 = \alpha b_1 + \beta b_2 + \gamma b_3$, $\lambda_3 = \alpha c_1 + \beta c_2 + \gamma c_3$ and $\lambda_4 = \alpha d_1 + \beta d_2 + \gamma d_3$. The above equation is a polynomial equation of degree 3 if $\lambda_1 \neq 0$, so it has three real roots t_1, t_2 and t_3. ($A, B \in \gamma$ implies that there are at least 2 real roots, so the third root is also real.) By Viéte's formulas we obtain that $\sum\limits_{i=1}^{3} t_i = -\dfrac{\lambda_2}{\lambda_1}$ and $t_1t_2 + t_1t_3 + t_2t_3 = \dfrac{\lambda_3}{\lambda_1}$. If $b_i = 0, i = 1,2,3$, it follows that $\lambda_2 = 0$, therefore we have the conclusion. Otherwise if $c_i = 0, i = 1,2,3$, it follows that $\lambda_3 = 0$. This implies that $t_1t_2 + t_1t_3 + t_2t_3 = 0$ and if we simplify with $t_1t_2t_3$ (if one of these roots is zero, then we don't have three intersection points) then we get $a+b+c=0$, where $a = 1/t_1$, $b = 1/t_2$ and $c = 1/t_3$. If $a+b=0$, the third intersection point is the point at infinity. □

Lemma 1.2. *If γ is a $C-$type curve and $A(a)$, $B(b)$, $C(c)$ are three points on γ, then the circle passing through A, B and C intersects γ at a fourth point $D(d)$ with the property $a+b+c+d=0$.*

Proof. Let $x^2+y^2+\alpha x+\beta y+\gamma=0$ be the equation of the circle $\mathcal{C}(ABC)$. For the intersection points of $\mathcal{C}(ABC)$ with γ we obtain

$$\frac{f_1^2(t) + f_2^2(t)}{g(t)} + \alpha f_1(t) + \beta f_2(t) + \gamma g(t) = 0. \tag{1.4}$$

By the assumptions $gr\frac{f_1^2+f_2^2}{g} = 4$ and the above equation has four real roots t_1, t_2, t_3 and t_4. From the given conditions we deduce that the coefficient of t in the left hand side of (1.4) is 0, so by Viéte's relations we obtain

$$\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4} = 0.$$

Changing the parametrization $\left(t \to \frac{1}{t}\right)$ and using our notations we have $a+b+c+d=0$. If $a+b+c=0$, we consider the point at infinity as the fourth intersection point. □

Remark 1.2. *In the previous lemmas the obtained conditions are necessary and sufficient for the collinearity (respectively the concyclicity) of three (four) points from the curve.*

Theorem 1.1. *If γ is an $L-$type curve the following statements are valid:*

a) *If the lines d_1, d_2 intersect γ in six points ($\{A_i, B_i, C_i\} = d_i \cap \gamma$, $i \in \{1, 2\}$) then the intersections of the lines A_1A_2, B_1B_2, C_1C_2 with γ are on a straight line;*

b) *If the line d intersects γ in three points ($\{A, B, C\} = d \cap \gamma$) then the intersections of the tangents in A, B, C with γ are on a straight line (here A, B, C are the tangency points);*

c) *If the line d intersects γ in three points ($\{A, B, C\} = d \cap \gamma$) then the intersections of the tangents in A, B, C with γ are on a straight line (here A, B, C are not the tangency points).*

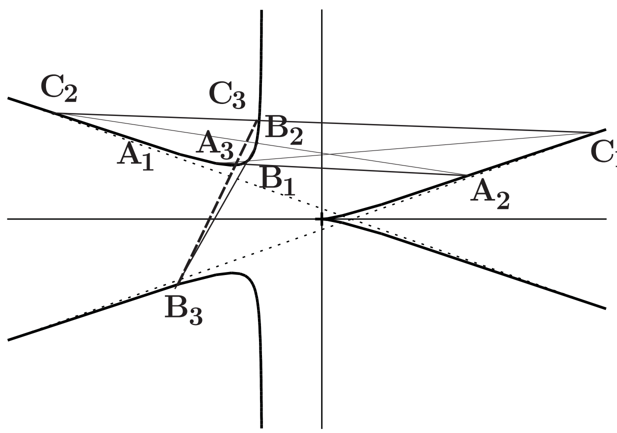

Figure 1.

Proof. a) Due to lemma 1.1 and the notations we've introduced we have the following relations: $a_1 + b_1 + c_1 = 0$, $a_2 + b_2 + c_2 = 0$. If we denote by A_3, B_3 and C_3 the intersection points, we have $a_1 + a_2 + a_3 = 0$, $b_1 + b_2 + b_3 = 0$ and $c_1 + c_2 + c_3 = 0$. Adding these relations therm by term and using the previous equalities, we obtain $a_3 + b_3 + c_3 = 0$. From remark 1.2 we deduce that A_3, B_3 and C_3 are on a straight line (see figure 1.).

b) Instead of considering tangents to γ in the points A, B and C we can consider $A_1 = A_2 = A$, $B_1 = B_2 = B$ and $C_1 = C_2 = C$ in the previous property. If we denote by A_3, B_3 and C_3 the intersection points of the tangents with the curve γ, we have $a_3 = -2a$, $b_3 = -2b$ and $c_3 = -2c$, so $a_3 + b_3 + c_3 = 0$ (see figure 2.).

c) If we consider the points $A_1(-a/2)$, $B_1(-b/2)$ and $C_1(-c/2)$, the tangents in A_1, B_1 and C_1 intersect γ in A, B respectively C. From $a + b + c = 0$ we obtain $-\frac{a}{2} - \frac{b}{2} - \frac{c}{2} = 0$, so the points A_1, B_1 and C_1 are on a straight line. (In fact we obtain the same figure as in b)).

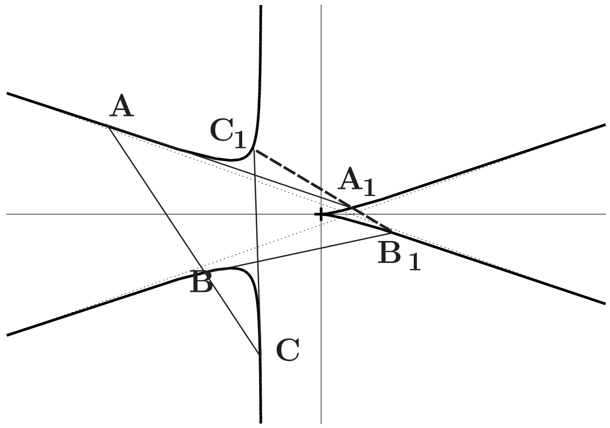

Figure 2.

□

Theorem 1.2. *If γ is a $C-$type curve the following properties are true:*

a) *If each of the circles $\mathcal{C}_i$, $i \in \{1,2,3\}$ intersects γ in four different points ($\{A_i, B_i, C_i, D_i\} = \mathcal{C}_i \cap \gamma, i \in \{1,2,3\}$) then the circles passing through the triplets (A_1, A_2, A_3), (B_1, B_2, B_3), (C_1, C_2, C_3), (D_1, D_2, D_3) intersect γ in four points situated on a circle;*
b) *If the circle $\mathcal{C}$, intersects γ in four different points $\{A, B, C, D\} = \mathcal{C} \cap \gamma$ then the osculating circles in these intersection points intersect γ in four points situated on a circle;*
c) *If the circle $\mathcal{C}$, intersects γ in four different points $\{A, B, C, D\} = \mathcal{C} \cap \gamma$ then there exist the points A_1, B_1, C_1, D_1 such that the osculating circles in A_1, B_1, C_1, D_1 passes through A, B, C respectively D and the points A_1, B_1, C_1, D_1 are situated on a circle;*
d) *If the line d intersects γ in three points then the osculating circles in these intersection points intersect γ in three points situated on a straight line;*
e) *If the circle $\mathcal{C}$, intersects γ in four different points $\{A, B, C, D\} = \mathcal{C} \cap \gamma$ then the tangents in these intersection points to γ intersect γ in four points situated on a circle;*
f) *If the circle $\mathcal{C}$, intersects γ in four different points $\{A, B, C, D\} = \mathcal{C} \cap \gamma$ then there exist the points A_1, B_1, C_1, D_1 such that the tangents in these points to γ intersect γ in A, B, C respectively D. Moreover the points A_1, B_1, C_1, D_1 are situated on a circle;*
g) *If the circle $\mathcal{C}$, intersects γ in four different points $\{A, B, C, D\} = \mathcal{C} \cap \gamma$ and we denote by E, F, G, H, I, J the intersection points of the lines AB, BC, CD, DA, AC respectively BD with γ then each of the quadruplets (E, F, G, H), (J, G, I, E), (F, H, J, I) is situated on a circle.*

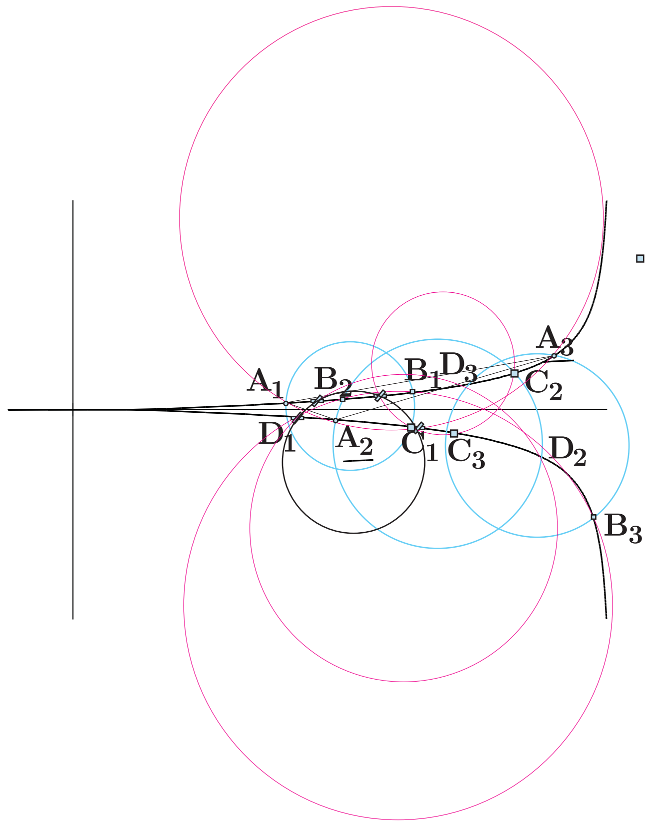

Figure 3.

Proof. a) From the construction and lemma 1.2 we have the following relations:

$$a_i + b_i + c_i + d_i = 0, \quad i \in \{1, 2, 3\}.$$

If we denote by A_4, B_4, C_4 and D_4 the fourth intersection point of the circles (A_1, A_2, A_3), (B_1, B_2, B_3), (C_1, C_2, C_3), respectively (D_1, D_2, D_3) with γ, we also have

$$a_1 + a_2 + a_3 + a_4 = 0, \ b_1 + b_2 + b_3 + b_4 = 0,$$

$$c_1 + c_2 + c_3 + c_4 = 0, \ d_1 + d_2 + d_3 + d_4 = 0.$$

Adding the last four relations term by term and using the first four, we deduce $a_4 + b_4 + c_4 + d_4 = 0$. Due to remark 1.2 the points A_4, B_4, C_4 and D_4 are on a circle (see figure 3.).

b) We apply the previous property for $A_1 = A_2 = A_3 = A$, $B_1 = B_2 = B_3 = B$, $C_1 = C_2 = C_3 = C$ and $D_1 = D_2 = D_3 = D$.

c) We consider the same figure as in the previous property, with changing the role of the points $\{A, B, C, D\}$ and $\{A_4, B_4, C_4, D_4\}$.

d) Let's denote by A, B, C the intersection points of d with γ and with A_1, B_1, C_1 the second intersection of the osculating circles with γ.

Due to lemma 1.2 and the definition of an osculating circle (as the limit of the circle passing through the points M, N, P on γ when these points tend to a fixed point on the curve) we have $a_1 = -3a$, $b_1 = -3b$ and $c_1 = -3c$. These equalities and lemma 1.1 imply that A_1, B_1 and C_1 are on a straight line.

e) If we denote by A_1, B_1, C_1 and D_1 the intersection points of the mentioned tangent lines with the curve, due to lemma 1.1 we have $a_1 = -2a$, $b_1 = -2b$, $c_1 = -2c$ and $d_1 = -2d$. From lemma 1.2 we deduce $a_1 + b_1 + c_1 + d_1 = 0$, so by remark 1.2 the points A_1, B_1, C_1 and D_1 are on a circle.

f) By changing the role of the points A, B, C, D and A_1, B_1, C_1, D_1 in the previous property we obtain the proof.

g) Due to lemma 1.1 we have $e = -a - b$, $f = -b - c$, $g = -c - d$, $h = -d - a$, $i = -a - c$ and $j = -b - d$, so we have $e + f + g + h = j + g + i + e = f + h + j + i = -2(a + b + c + d) = 0$. This completes the proof.

$\square$

In the last part we give some analogues of the above definitions, lemmas and theorems in higher dimensions. The proofs are obvious and they are left to the reader.

Definition 1.3. *The curve γ is called a $P-$type curve if admits a parametric representation of the form*

$$\left\{ \begin{array}{l} x = \frac{f_1(t)}{g(t)} \\ y = \frac{f_2(t)}{g(t)} \\ z = \frac{f_3(t)}{g(t)} \end{array} \right. \tag{1.5}$$

where the coefficients of the functions $f_i(t) = a_i t^4 + b_i t^3 + c_i t^2 + d_i t_i + e_i$, $i \in \{1, 2, 3\}$, $g(t) = a_4 t^4 + b_4 t^3 + c_4 t^2 + d_4 t + e_4$ satisfy the following condition

$$b_1 = b_2 = b_3 = b_4 = 0 \text{ or } d_1 = d_2 = d_3 = d_4 = 0.$$

Definition 1.4. *The curve γ is called an $S-$type curve if admits a parametric representation of the form*

$$\left\{ \begin{array}{l} x = \frac{f_1(t)}{g(t)} \\ y = \frac{f_2(t)}{g(t)} \\ z = \frac{f_3(t)}{g(t)} \end{array} \right. \tag{1.6}$$

where the functions $f_i(t) = a_i t^4 + b_i t^3 + c_i t^2 + e_i$, $i \in \{1, 2, 3\}$, $g(t) = b_4 t^3 + c_4 t^2 + e_4$ satisfy the following condition

$$\left(f_1^2 + f_2^2 + f_3^2\right) \vdots g \text{ in } \mathbb{R}[X]. \tag{1.7}$$

Remark 1.3. 1. *It is obvious that any $S-$type curve is an $P-$type curve;*

2. *The analogous of the straight cissoid defined by the equations* $x(t) = \frac{rt^2}{1+t^3}$, $y(t) = \frac{rt^3}{1+t^3}$ *and* $z(t) = \frac{rt^4}{1+t^3}$ *is an* $S-$*type curve. The curve defined by the relations* $x(t) = \frac{rt^2}{1+t^4}$, $y(t) = \frac{rt^3}{1+t^4}$ *and* $z(t) = \frac{rt^4}{1+t^4}$ *is a* $P-$*type curve but it is not an* $S-$*type curve.*

The main tools in establishing our results are the following two lemmas:

Lemma 1.3. *If* γ *is a* $P-$*type curve and* $A(a)$, $B(b)$ *and* $C(c)$ *are three points on* γ, *then the plane* ABC *intersects* γ *at a fourth point* $D(d)$ *with the property* $a + b + c + d = 0$.

Lemma 1.4. *If* γ *is an* $S-$*type curve and* $A(a)$, $B(b)$, $C(c)$ *and* $D(d)$ *are four points on* γ, *then the sphere passing through* A, B, C *and* D *intersects* γ *at a fifth point* $E(e)$ *with the property* $a + b + c + d + e = 0$.

Theorem 1.3. *If* γ *is a* $P-$*type curve the following statements are valid:*

a) *If the planes* α_1, α_2 *and* α_3 *intersect* γ *in four points* ($\{A_i, B_i, C_i, D_i\} = \alpha_i \cap \gamma$, $i \in \{1, 2, 3\}$) *then the intersections of the planes* $A_1A_2A_3$, $B_1B_2B_3$, $C_1C_2C_3$ *and* $D_1D_2D_3$ *with* γ *are in a plane;*

b) *If the plane* α *intersects* γ *in four points* ($\{A, B, C, D\} = d \cap \gamma$) *then the intersections of the tangent planes in* A, B, C *and* D *with* γ *are in a plane (here* A, B, C *and* D *are the tangency points);*

c) *If the plane* α *intersects* γ *in four points* ($\{A, B, C, D\} = d \cap \gamma$) *then the intersections of the tangent planes in* A, B, C *and* D *with* γ *are in a plane (here* A, B, C *and* D *are not the tangency points).*

Theorem 1.4. *If* γ *is an* $S-$*type curve the following properties are true:*

a) *If each of the spheres* $\mathcal{S}_i$, $i \in \{1, 2, 3, 4\}$ *intersects* γ *in five different points,* ($\{A_i, B_i, C_i, D_i, E_i\} = \mathcal{S}_i \cap \gamma$, $i \in \{1, 2, 3\}$) *then the spheres passing through the quadruplets* (A_1, A_2, A_3, A_4), (B_1, B_2, B_3, B_4), (C_1, C_2, C_3, C_4),
(D_1, D_2, D_3, D_4) *and* (E_1, E_2, E_3, E_4) *intersect* γ *in five points situated on a sphere;*

b) *If the sphere* $\mathcal{S}$, *intersects* γ *in five different points,* ($\{A, B, C, D, E\} = \mathcal{S} \cap \gamma$) *then the osculating spheres in these intersection points intersect* γ *in five points situated on a sphere;*

c) *If the sphere* $\mathcal{S}$, *intersects* γ *in five different points,* ($\{A, B, C, D, E\} = \mathcal{S} \cap \gamma$) *then there exists the points* A_1, B_1, C_1, D_1, E_1 *such that the osculating spheres in* A_1, B_1, C_1, D_1, E_1 *passes through* A, B, C, D *respectively* E *and the points* A_1, B_1, C_1, D_1, E_1 *are situated on a sphere;*

d) *If the plane* α *intersects* γ *in four points, then the osculating spheres in these intersection points intersect* γ *in four points situated in a plane;*

e) *If the sphere* $\mathcal{S}$, *intersects* γ *in five different points,* ($\{A, B, C, D, E\} = \mathcal{S} \cap \gamma$) *then the tangent planes in these intersection points to* γ *intersect* γ *in five points situated on a sphere;*

f) *If the sphere $\mathcal{S}$, intersects γ in five different points, ($\{A, B, C, D, E\} = \mathcal{S} \cap \gamma$) then there exist the points A_1, B_1, C_1, D_1, E_1 such that the tangent planes in these points to γ intersect γ in A, B, C, D respectively E. Moreover the points A_1, B_1, C_1, D_1, E_1 are situated on a sphere;*

g) *If the sphere $\mathcal{S}$, intersects γ in five different points, ($\{A, B, C, D, E\} = \mathcal{S} \cap \gamma$) and we denote by $F, G, H, I, J, K, L, M, N, O$ the intersection points of the planes ABC, ABD, ABE, ACD, ACE, ADE, BCD, BCE, BDE respectively CDE with γ, then among these intersection points there exist $\frac{4!}{2} = 12$ quintet (for example (F, L, O, K, H), (I, N, J, G, M), (F, G, J, N, O) and (F, H, I, M, N)) such that each quintet is situated on a sphere.*

Definition 1.5. *The curve γ is called an $HP-$type curve if admits a parametric representation of the form*

$$\left\{\begin{array}{l} x_1 = \frac{f_1(t)}{g(t)} \\ x_2 = \frac{f_2(t)}{g(t)} \\ \cdots\cdots\cdots \\ x_n = \frac{f_n(t)}{g(t)} \end{array}\right. \tag{1.8}$$

where the coefficients of the functions $f_i(t) = \sum\limits_{j=0}^{n+1} a_{ij} t^j$, $i \in \{1, 2, 3, \cdots, n\}$, $g(t) = \sum\limits_{j=0}^{n+1} a_{(n+1)j} t^j$ satisfy the following condition

$a_{i1} = 0$ *for all* $i \in \{1, 2, \cdots, n+1\}$ *or* $a_{in} = 0$ *for all* $i \in \{1, 2, \cdots, n+1\}$.

Definition 1.6. *The curve γ is called an $HS-$type curve if admits a parametric representation of the form*

$$\left\{\begin{array}{l} x_1 = \frac{f_1(t)}{g(t)} \\ x_2 = \frac{f_2(t)}{g(t)} \\ \cdots\cdots\cdots \\ x_n = \frac{f_n(t)}{g(t)} \end{array}\right. \tag{1.9}$$

where the functions $f_i(t) = \sum\limits_{\substack{j=0,\\ j\neq 1}}^{n+1} a_{ij} t^j$, $i \in \{1, 2, 3, \cdots, n\}$ and $g(t) = \sum\limits_{\substack{j=0,\\ j\neq 1}}^{n} a_{(n+1)j} t^j$ satisfy the following condition

$$\left(\sum_{i=1}^{n} f_i^2\right) \vdots g \text{ in } \mathbb{R}[X]. \tag{1.10}$$

Remark 1.4. 1. *It is obvious that any $HS-$type curve is an $HP-$type curve;*

2. *The analogous of the straight cissoid defined by the equations $x_i(t) = \dfrac{rt^{i+1}}{1+t^n}$, for all $i \in \{1, 2, \cdots, n\}$ is an $HS-$type curve. The curve*

defined by the relations $x_i(t) = \dfrac{rt^{i+1}}{1+t^{n+1}}$, *for all* $i \in \{1, 2, \cdots, n\}$ *is an* $HP-$*type curve but it is not a* $HS-$*type curve.*

The main tools in establishing our results are the following two lemmas:

Lemma 1.5. *If* γ *is an* $HP-$*type curve and* $A_i(a_i)$, $i \in \{1, 2, \cdots, n\}$ *are* n *points on* γ, *then the hyperplane* $A_1A_2...A_n$ *intersects* γ *at an* $(n+1)^{th}$ *point* $A_{n+1}(a_{n+1})$ *with the property* $\sum\limits_{i=1}^{n+1} a_i = 0$.

Lemma 1.6. *If* γ *is an* $HS-$*type curve and* $A_i(a_i)$, $i \in \{1, 2, \cdots, n+1\}$ *are* $n+1$ *points on* γ, *then the hypersphere passing through* $A_1A_2...A_{n+1}$ *intersects* γ *at an* $(n+2)^{th}$ *point* $A_{n+2}(a_{n+2})$ *with the property* $\sum\limits_{i=1}^{n+2} a_i = 0$.

Theorem 1.5. *If* γ *is an* $HP-$*type curve the following statements are valid:*

a) *If the hyperplanes* α_i, $i \in \{1, 2, \cdots, n\}$ *intersect* γ *in* $n+1$ *points, (*$\{A_{ij} | j \in \{1, 2, \cdots, n+1\} = \alpha_i \cap \gamma$, $i \in \{1, 2, \cdots, n\}$*) then the intersections of the hyperplanes* $(A_{ij})_{1 \le i \le n}$ *with* γ *are in a hyperplane;*

b) *If the hyperplane* α *intersects* γ *in* $n+1$ *points, (*$\{A_j | j \in \{1, 2, \cdots, n+1\}\} = \alpha \cap \gamma$*) then the intersections of the tangent hyperplanes in these intersection points with* γ *are in a hyperplane (here* $(A_j)_{1 \le j \le n+1}$ *are the tangency points);*

c) *If the hyperplane* α *intersects* γ *in* $n+1$ *points, (*$\{A_j | j \in \{1, 2, \cdots, n+1\}\} = \alpha \cap \gamma$*) then the intersections of the tangent hyperplanes in these intersection points with* γ *are in a hyperplane (here* $(A_j)_{1 \le j \le n+1}$ *are not the tangency points);*

Theorem 1.6. *If* γ *is an* $HS-$*type curve the following properties are true:*

a) *If each of the hyperspheres* $\mathcal{S}_i$, $i \in \{1, 2, \cdots, n+1\}$ *intersect* γ *in* $n+2$ *different points, (*$\{A_{ij} | j \in \{1, 2, \cdots, n+2\} = \mathcal{S}_i \cap \gamma$, $i \in \{1, 2, \cdots, n+1\}$*) then the hyperspheres passing through the* $(n+1)-$*tuples* $(A_{ij})_{1 \le i \le n+1}$ *intersect* γ *in* $n+2$ *points situated on a hypersphere;*

b) *If the hypersphere* $\mathcal{S}$, *intersects* γ *in* $n+2$ *different points, (*$\{A_j | j \in \{1, 2, \cdots, n+2\} = \mathcal{S} \cap \gamma$*) then the osculating hyperspheres in these intersection points intersect* γ *in* $n+2$ *points situated on a hypersphere;*

c) *If the hypersphere* $\mathcal{S}$, *intersects* γ *in* $n+2$ *different points, (*$\{A_j | j \in \{1, 2, \cdots, n+2\} = \mathcal{S} \cap \gamma$*) then there exists the points* A'_j, *where* $j \in \{1, 2, \cdots, n+2\}$ *such that the osculating hyperspheres in these points passes through the initially intersection points. Moreover these points are situated on a hypersphere;*

d) *If the hyperplane* α *intersects* γ *in* $n+1$ *points, then the osculating hyperspheres in these intersection points intersect* γ *in* $n+1$ *points situated in a hyperplane;*

e) *If the hypersphere* $\mathcal{S}$, *intersects* γ *in* $n+2$ *different points,* ($\{A_j | j \in \{1,2,\cdots,n+2\} = \mathcal{S} \cap \gamma$) *then the tangent hyperplanes in these intersection points to* γ *intersect* γ *in* $n+2$ *points situated on a hypersphere;*

f) *If the hypersphere* $\mathcal{S}$, *intersects* γ *in* $n+2$ *different points,* ($\{A_j | j \in \{1,2,\cdots,n+2\} = \mathcal{S} \cap \gamma$) *then there exist the points* A'_j, *where* $j \in \{1,2,\cdots,n+2\}$, *such that the tangent hyperplanes in these points to* γ *intersect* γ *in* $A_1, A_2, \cdots A_{n+2}$. *Moreover these points are situated on a hypersphere;*

g) *If the hypersphere* $\mathcal{S}$, *intersects* γ *in* $n+2$ *different points* ($\{A_j | j \in \{1,2,\cdots,n+2\} = \mathcal{S} \cap \gamma$) *and we denote by* M_{ij} *the intersection of the hyperplane determined by the points* $(A_k)_{\substack{1\leq k\leq n+2,\\ i\neq k\neq j}}$ *with* γ *for* $i,j \in \{1,2,\cdots,n+2\}$, $i \neq j$, *then for every cyclic permutation*

$$\begin{pmatrix} 1 & 2 & \cdots & n+2 \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n+2) \end{pmatrix}$$

the points $(M_{i\sigma(i)})_{1\leq i\leq n+2}$ *are situated on a hypersphere.*

If we use the first and the last coefficient in the Viéte's relations, we can obtain an other class of curves:

Definition 1.7. *The curve* γ *is called an* $HPL-$*type curve if admits a parametric representation of the form*

$$\left\{\begin{array}{l} x_1 = \frac{f_1(t)}{g(t)} \\ x_2 = \frac{f_2(t)}{g(t)} \\ \ldots\ldots\ldots\ldots \\ x_n = \frac{f_n(t)}{g(t)} \end{array}\right. \tag{1.11}$$

where the functions $f_i(t) = \sum\limits_{j=0}^{n+1} a_{ij}t^j$, $i \in \{1,2,3,\cdots,n\}$ *and* $g(t) = \sum\limits_{j=0}^{n+1} a_{(n+1)j}t^j$ *satisfy the following condition*

$$a_{i(n+1)} = (-1)^{n+1}a_{i0},\ \ i \in \{1,2,3,...,n+1\} \tag{1.12}$$

or n *is even and* $a_{i(n+1)} = a_{i0}$, $i \in \{1,2,3,...,n+1\}$.

Theorem 1.7. *If* γ *is a* $HPL-$ *type curve, the properties from theorem 1.5 are valid.*

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